

Introduction to Vibrations

Free Response Part 1: Spring-mass systems

Vibration is a sub-discipline of dynamics that deals with repetitive motions. Some familiar examples are the vibrations of automobiles, guitar strings, cell phones and pendulums. Vibrations can be unwanted or wanted. For example, vibrations in automobiles and aircrafts are undesired because they can result in discomfort to the passengers and structural damage due to fatigue. However, in the case of music and communication, vibrations are extremely useful. Vibrations are modeled mathematically using fundamental principles and analyzed using results from calculus and differential equations. In this module, we will only consider single-degree-of-freedom models where there is only one displacement coordinate. We begin by studying the motion of a spring-mass system and, in the next module, continue by discussing a spring-mass system with damping.

Spring-Mass Model

Consider a mass attached to a massless spring on a frictionless surface, as shown below.

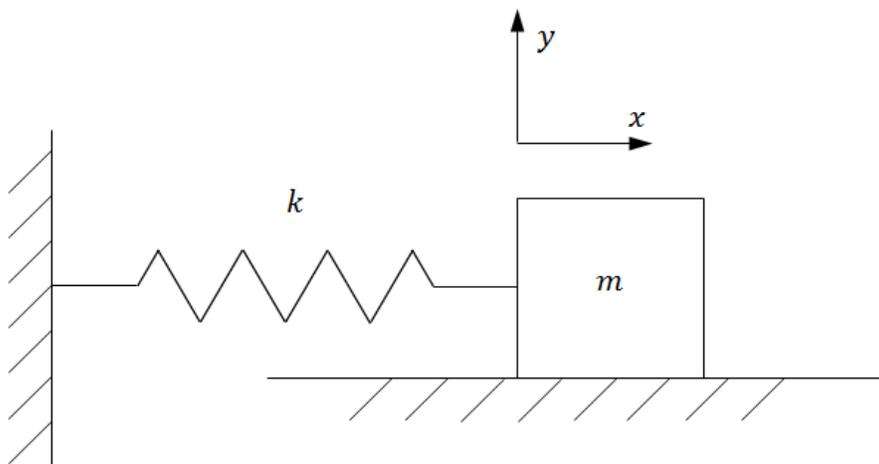


Fig. 1: Spring-mass system

Here, k is the stiffness of the spring (also known as the spring constant) and m is the mass of the object. The force f_k applied by the spring, which "tries" to restore the spring to

an unstretched state, is given by

$$f_k = -k \cdot x$$

... Eq. (1)

where x is the displacement measured from the unstretched position of the spring. Summing the forces on the mass in the x -direction yields

$$m \cdot \ddot{x}(t) = -k \cdot x(t) \quad \text{or} \quad m \cdot \ddot{x}(t) + k \cdot x(t) = 0$$

... Eq. (2)

where $\ddot{x}(t)$ is the acceleration. This equation is a second order linear homogeneous ordinary differential equation with constant coefficients and can be solved by assuming that the solution is of the form

$$x(t) = a \cdot e^{\lambda \cdot t}$$

... Eq. (3)

where a and λ are two unknown non-zero constants. Differentiating $x(t)$ twice yields

$$\dot{x}(t) = \lambda \cdot a \cdot e^{\lambda \cdot t}$$

... Eq. (4)

and

$$\ddot{x}(t) = \lambda^2 \cdot a \cdot e^{\lambda \cdot t}$$

... Eq. (5)

Substituting Eqs. (4) and (5) into Eq. (2) yields

$$m \cdot \lambda^2 \cdot a \cdot e^{\lambda \cdot t} + k \cdot a \cdot e^{\lambda \cdot t} = 0$$

... Eq. (6)

Since a and λ are non-zero constants, this equation reduces to

$$m \cdot \lambda^2 + k = 0$$

... Eq. (7)

This equation is also known as the characteristic equation. Solving for λ gives

$$\lambda = \pm \sqrt{-\frac{k}{m}} = \pm \sqrt{\frac{k}{m}} j$$

... Eq. (8)

where $j = \sqrt{-1}$. This shows that there are two solutions for the differential equation

$$x(t) = a_1 \cdot e^{\sqrt{\frac{k}{m}} j \cdot t} \quad \text{and} \quad x(t) = a_2 \cdot e^{-\sqrt{\frac{k}{m}} j \cdot t}$$

... Eqs. (9a) and (9b)

Since Eq. (2) is linear, the solution of the equation can be written as a linear combination of these two solutions:

$$x(t) = a_1 \cdot e^{\sqrt{\frac{k}{m}} j \cdot t} + a_2 \cdot e^{-\sqrt{\frac{k}{m}} j \cdot t}$$

... Eq. (10)

Using Euler's identity ($e^{j \cdot \alpha} = \cos(\alpha) + j \cdot \sin(\alpha)$), we can rewrite this as

$$x(t) = A \cdot \sin\left(\sqrt{\frac{k}{m}} t + \phi\right)$$

... Eq. (11)

where A and ϕ are constants that can be determined using the initial conditions of the mass. This equation shows that the solution to the differential equation that describes the motion of the mass is sinusoidal with a frequency (in rad/s) equal to $\sqrt{\frac{k}{m}}$. This frequency is called the natural frequency of the system and is represented by ω_n . It is important to note that this frequency is independent of the initial conditions as long as k and m are constant. Eq. (11) can now be rewritten as

$$x(t) = A \cdot \sin(\omega_n t + \phi)$$

... Eq. (12)

The constants A and ϕ are dependant on the initial conditions. With the following equations for the initial displacement and velocity

$$x_0 = x(0) = A \cdot \sin(0 + \phi)$$

... Eq. (13)

and

$$v_0 = v(0) = A \cdot \omega_n \cdot \cos(0 + \phi)$$

... Eq. (14)

we can obtain the following equations for the two unknowns A and ϕ

$$A = \frac{\sqrt{\omega_n^2 \cdot x_0^2 + v_0^2}}{\omega_n}$$

... Eq. (15)

and

$$\phi = \tan^{-1} \left(\frac{\omega_n \cdot x_0}{v_0} \right)$$

... Eq. (16)

While using Eq. (12) it is important to ensure that the phase angle calculated using Eq. (16) corresponds to the correct quadrant. For example if x_0 is positive and v_0 is negative, then ϕ lies in the second quadrant. However, most software and calculators will return a value that lies between $-\pi/2$ and $\pi/2$, so it may be required to correct the angle or use a function that takes the signs of the numerator and denominator into account.

By differentiating Eq. (12) we get the following equation for the velocity:

$$\dot{x}(t) = A \cdot \omega_n \cdot \cos(\omega_n \cdot t + \phi)$$

... Eq. (17)

And by differentiating it again we get the following equation for the acceleration:

$$\ddot{x}(t) = -A \cdot \omega_n^2 \cdot \sin(\omega_n \cdot t + \phi)$$

... Eq. (18)

Eq. (17) shows that the velocity, $\dot{x}(t)$, is 90° or $\pi/2$ radians out of phase with the position [using the identity $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$] and Eq. (18) shows that the acceleration, $\ddot{x}(t)$, is

180 degrees out of phase with the position.

Example 1: Overlay of position, velocity and acceleration for a spring-mass system

The mass of a spring-mass system is 1 kg and the stiffness of the spring is 2 N/m. If the mass is displaced such that the spring is extended by 0.1m (positive x-direction) and is given an initial velocity of 0.1m/s towards its equilibrium position (negative x-direction), plot the displacement, velocity and acceleration of the mass for 10 seconds.

restart :

$$m := 1 : \quad [\text{kg}]$$

$$k := 2 : \quad \left[\frac{\text{N}}{\text{m}} \right]$$

$$x_0 := 0.1 : \quad [\text{m}]$$

$$v_0 := -0.1 : \quad \left[\frac{\text{m}}{\text{s}} \right]$$

$$\omega_n := \sqrt{\frac{k}{m}} = \sqrt{2} \quad \left[\frac{\text{rad}}{\text{s}} \right]$$

Using Eqs. (15) and (16) we get

$$A := \frac{\sqrt{\omega_n^2 \cdot x_0^2 + v_0^2}}{\omega_n} :$$

and

$$\phi_I := \tan^{-1} \left(\frac{\omega_n \cdot x_0}{v_0} \right) = -\arctan(1.000000000 \sqrt{2}) \xrightarrow{\text{at 5 digits}} -0.95531$$

Since the numerator of the argument of the inverse tan function is positive and the denominator is negative, the required angle should be in the second quadrant. Hence Pi rad must be added to this angle.

$$\phi := \phi_l + \text{Pi} = -\arctan(1.000000000 \cdot \sqrt{2}) + \pi \xrightarrow{\text{at 5 digits}} 2.1863$$

The following are the three functions for the displacement, velocity and acceleration of the mass.

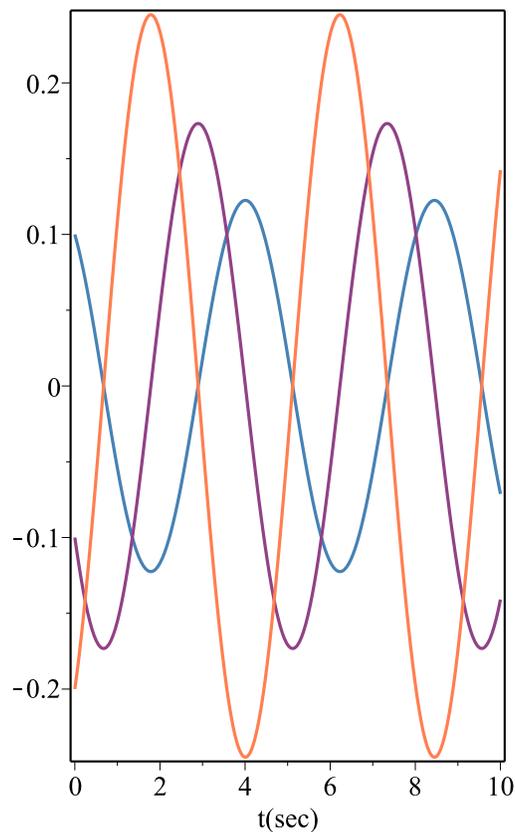
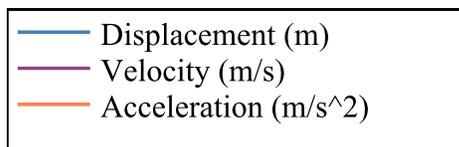
$$x := t \rightarrow A \cdot \sin(\omega_n \cdot t + \phi) :$$

$$v := t \rightarrow A \cdot \omega_n \cdot \cos(\omega_n \cdot t + \phi) :$$

$$a := t \rightarrow -A \cdot \omega_n^2 \cdot \sin(\omega_n \cdot t + \phi) :$$

The following plot shows an overlay of the displacement, velocity and acceleration of the mass. No unit has been specified on the y-axis because all three curves have different units as shown in the legend.

```
plot([x(t), v(t), a(t)], t=0..10, color=["SteelBlue", "Niagara DarkOrchid", "Coral"], labels
= ["t(sec)", ""], legend=["Displacement (m)", "Velocity (m/s)", "Acceleration (m/s^2)"],
legendstyle=[location=left], axes=boxed);
```



This plot allows us to observe the phase differences between the three curves. When the displacement reaches its peak, the velocity is zero and the acceleration is minimum (that is, of maximum magnitude in the negative direction).

Example 2: Amplitude Surface-Plot

Make a three-dimensional surface plot of the amplitude of an undamped oscillator versus the initial conditions, x_0 and v_0 , for the range given by $-0.2 < x_0 < 0.2$ m and $-2 < v_0 < 2$ m/s, for a system with natural frequency of 10 rad/s.

restart :

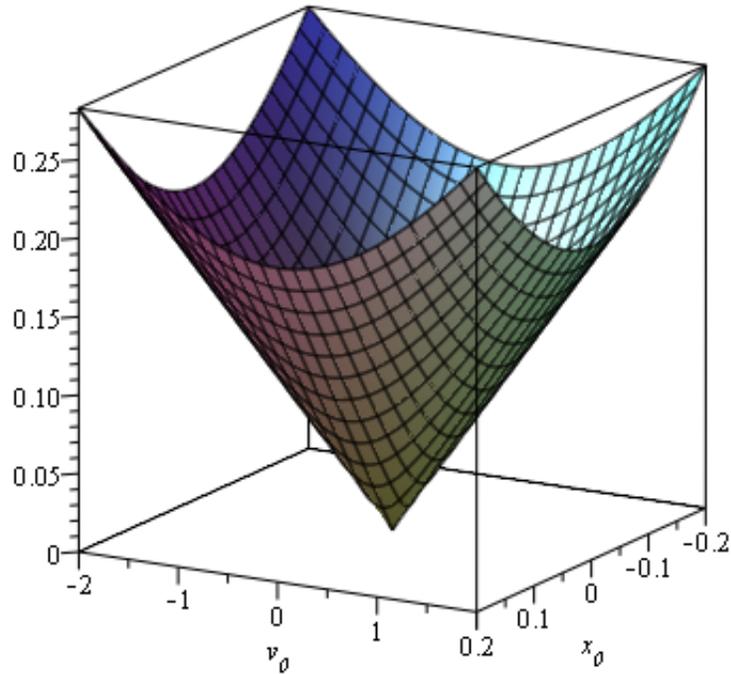
$$\omega_n := 10 : \quad \left[\begin{array}{l} \text{rad/s} \\ \end{array} \right]$$

Using Eq. (15)

$$A := \frac{\sqrt{\omega_n^2 \cdot x_0^2 + v_0^2}}{\omega_n} :$$

plot3d(A, x0=-0.2..0.2, v0=-2..2, axes=boxed, title=Amplitude vs initial conditions, orientation=[30, 75, 0])

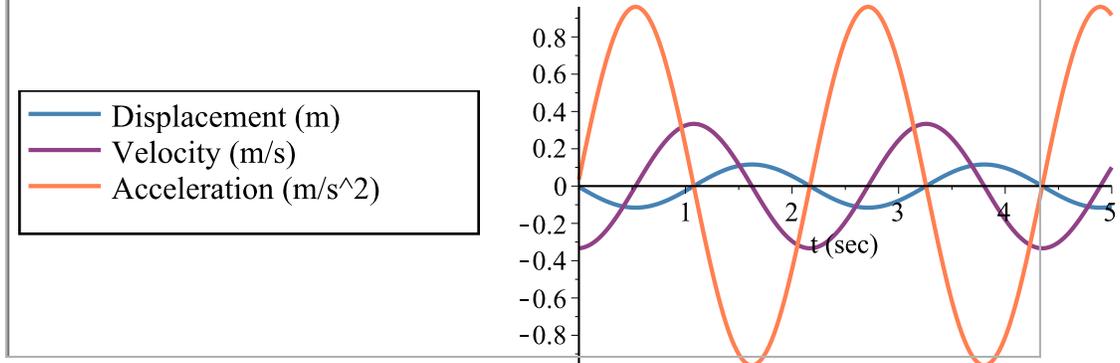
Amplitude vs initial conditions



This plot allows us to see the resulting amplitude of the displacement for different combinations of initial conditions.

For the following plot, the initial conditions and parameters of a spring-mass system can be varied using the gauges to see the effect on the displacement, velocity and acceleration of the mass.

Displacement, Velocity and Acceleration



Spring-mass analogs

Any other system that results in a differential equation of motion in the same form as Eq. (2) will show a response similar to the response of a spring-mass system. The following are a few examples of such single degree of freedom systems.

Table 1: Examples of systems analogous to a spring-mass system

<p style="text-align: center;">Spring and Mass $m \cdot \ddot{x} + k \cdot x = 0$</p>	<p style="text-align: center;">Shaft and disk $J \cdot \ddot{\theta} + k \cdot \theta = 0$</p>	<p style="text-align: center;">Simple pendulum $\ddot{\theta} + \left(\frac{g}{l} \right) \cdot \theta = 0$</p>

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_n = \sqrt{\frac{k}{J}}$$

$$\omega_n = \sqrt{\frac{g}{l}}$$

There are also many other such cases. For example, if an ideal charged capacitor is connected in parallel with an ideal inductor, the charge will oscillate with a natural frequency $\omega_n = \sqrt{\frac{1}{L \cdot C}}$ (where L is the inductance and C is the capacitance). This is why the spring-mass system model is a very commonly used system to study vibrations and also why many systems are simplified and modeled using equivalent springs and masses.

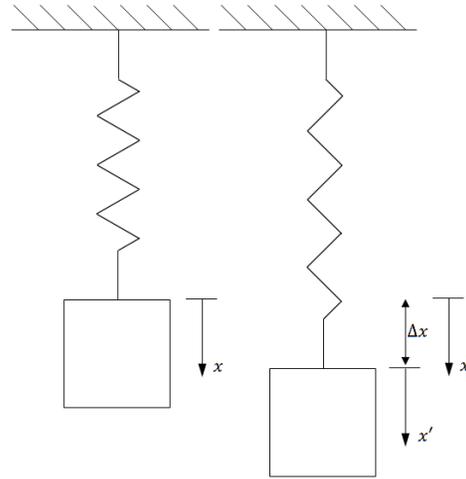
The effect of gravity for a hanging spring-mass system

An interesting question that arises while analyzing spring-mass systems is how does gravity effect the response of the system if the mass is hanging?

The answer is that it does not effect the equation for the displacement, the velocity and the acceleration. It only leads to a change in the mean position about which the vibration takes place. For the spring-mass system discussed above, the mean position about which the vibration takes place is the unstretched position of the spring. In the case of a hanging mass, the vibration takes place about the position of the mass when it is in static equilibrium.

Consider the same spring and mass described in the previous sections suspended vertically, as shown in Fig. 4. If x measures the vertical displacement with respect to the unstretched position, let us define another variable x' which measures the displacement with respect to the stretched equilibrium position such that,

$$x = x' + \Delta x \quad \dots \text{Eq. (19)}$$



In the stretched equilibrium position, since the weight of the mass is supported by the spring, we get

$$m \cdot g = k \cdot \Delta x \quad \dots \text{Eq. (20)}$$

Fig. 4: Hanging spring-mass: Unstretched and stretched positions.

Next, taking the sum of the forces in the vertical direction we get

$$m \cdot \ddot{x}(t) = m \cdot g - k \cdot x(t) \quad \dots \text{Eq. (21)}$$

Using Eq. (20) we can write this as

$$m \cdot \ddot{x}(t) + k \cdot (x(t) - \Delta x) = 0 \quad \dots \text{Eq. (22)}$$

By combining this equation with Eq. (19) we get

$$m \cdot \ddot{x}(t) + k \cdot (x'(t)) = 0 \quad \dots \text{Eq. (23)}$$

and, finally, since $\ddot{x}(t) = \ddot{x}'(t)$ (differentiating both sides of Eq. (19) twice with respect to time), we get

$$m \cdot \ddot{x}'(t) + k \cdot (x'(t)) = 0 \quad \dots \text{Eq. (24)}$$

which is in the same form as Eq. (2). Hence the vibration about the stretched equilibrium position will be identical to the vibration of a horizontal spring-mass about the unstretched position (ignoring friction of course).

An Example with MapleSim

Example 3: Spring-Pendulum

Problem Statement: A component of a machine is modeled as a pendulum connected to a spring (as shown in Fig. 5). Derive the equation of motion and find the natural frequency of the system. Also, find the amplitude of the angular velocity if the initial angular displacement is 0.175 rad (approx. 10°) measured from the vertical and if there is no initial angular velocity.

The mass of the pendulum is 2kg, the length of the pendulum is 0.5m and the stiffness of the spring is 20 N/m. Assume that the rod of the pendulum has no mass.

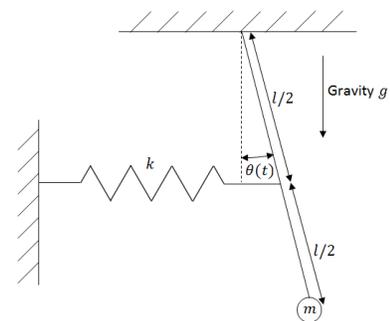


Fig. 5: Spring-pendulum example

Analytical Solution

Data:

restart :

$m := 2 :$ [kg]

$l := 0.5 :$ [m]

$k := 20 :$ [N/m
]

$$g := 9.81 : \quad \left[\frac{\text{m}}{\text{s}^2} \right]$$

$$\theta_0 := 0.175 : \quad [\text{rad}]$$

$$\omega_0 := 0 : \quad \left[\frac{\text{rad}}{\text{s}} \right]$$

Solution:

Since the angle is small, we will assume that the spring stretches and compresses in the horizontal direction only. Therefore, the force due to the spring can be written as

$$F_k := -k \cdot \left(\frac{l}{2} \right) \cdot \theta(t) :$$

Taking the sum of the moments of force about the pivot we get

$$J_0 \cdot \ddot{\theta}(t) = F_k - m \cdot g \cdot l \cdot \theta(t) :$$

where J_0 is the moment of inertia of the pendulum. This equation can be rewritten as

$$m \cdot l^2 \cdot \ddot{\theta}(t) + \left(k \cdot \left(\frac{l}{2} \right)^2 + m \cdot g \cdot l \right) \cdot \theta(t) = 0 :$$

or

$$m \cdot \ddot{\theta}(t) + \left(\frac{k}{4} + \frac{m \cdot g}{l} \right) \cdot \theta(t) = 0 :$$

Comparing this equation to the form of Eq. (2) we can conclude that the natural frequency of the system is

$$\omega_n := \sqrt{\frac{\frac{k}{4} + \frac{m \cdot g}{l}}{m}} = 4.70$$

Therefore, the natural frequency of the system is 4.70 rad/s.

The amplitude of the angular velocity can be found using Eqs. (14) and (15). Since the initial velocity is 0, the amplitude of the angular displacement is the initial angular displacement and the amplitude of the angular velocity is

$$\theta_0 \cdot \omega_n = .82$$

Therefore, the amplitude of the angular velocity is 0.82 rad/s.

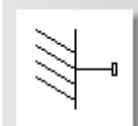
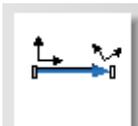
MapleSim Simulation

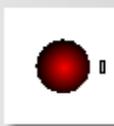
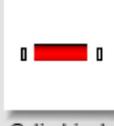
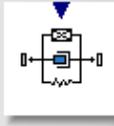
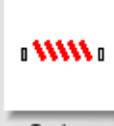
Constructing the model

Step1: Insert Components

Drag the following components into the workspace:

Table 2: Components and locations

Component	Location
 Fixed Frame	Multibody > Bodies and Frames
(2 required)	
 Revolute	Multibody > Joints and Motions
 Rigid Body Frame	Multibody > Bodies and Frames
(2 required)	
 Rigid Body	Multibody > Bodies and Frames

 <p>Spherical Geometry</p>	Multibody > Visualization
 <p>Path Trace</p>	Multibody > Visualization
 <p>Cylindrical Geometry</p>	Multibody > Visualization
 <p>Translational Spring Damper Actuator</p>	Multibody > Forces and Moments
 <p>Spring Geometry</p>	Multibody > Visualization

Step 2: Connect the components

Connect the components as shown in the following diagram (the dashed boxes are not part of the model, they have been drawn on top to help make it clear what the different components are for).

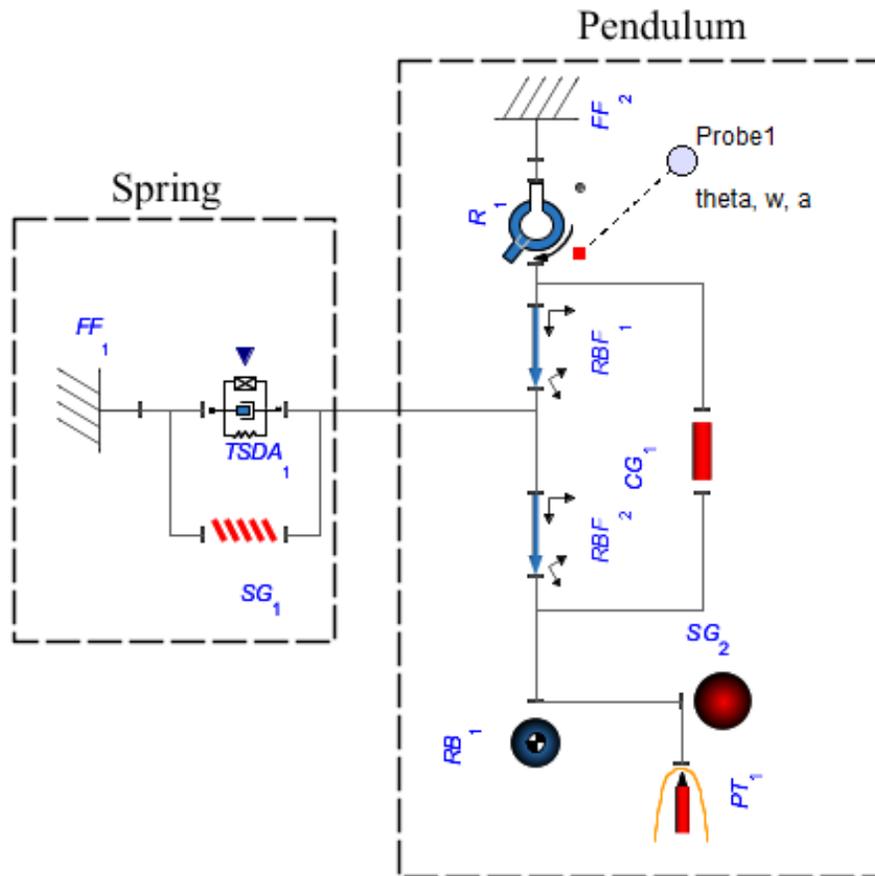


Fig. 6: Component diagram

Step 3: Set up the Pendulum

1. Click the **Revolute** component, enter **0.175 rad** for the initial angle (θ_0) and select **Strictly Enforce** in the drop down menu for the initial conditions ($IC_{\theta, \omega}$).
The axis of rotation (\hat{e}_1) should be left as the default axis **[0,0,1]**.
2. Enter **[0,-0.25,0]** for the x,y,z offset (\vec{r}_{XYZ}) of both the **Rigid Body Frames**.
3. Enter **2 kg** for the mass (m) of the rigid body frame.

Step 4: Set up the Spring

1. Click the **Fixed Frame** component connected to the **TSDA** (FF_1 in the

diagram) and enter **[-0.25,-0.25,0]** for the x,y,z offset (\vec{r}_{XYZ}).

2. Click the **TSDA** component and enter **20 N/m** for the spring constant (K_{spring}). Also, enter **0.25 m** for the unstretched length (l_0) to correspond to the location of the **Fixed Frame**. Leave the damping constant (K_{damper}) as **0**.

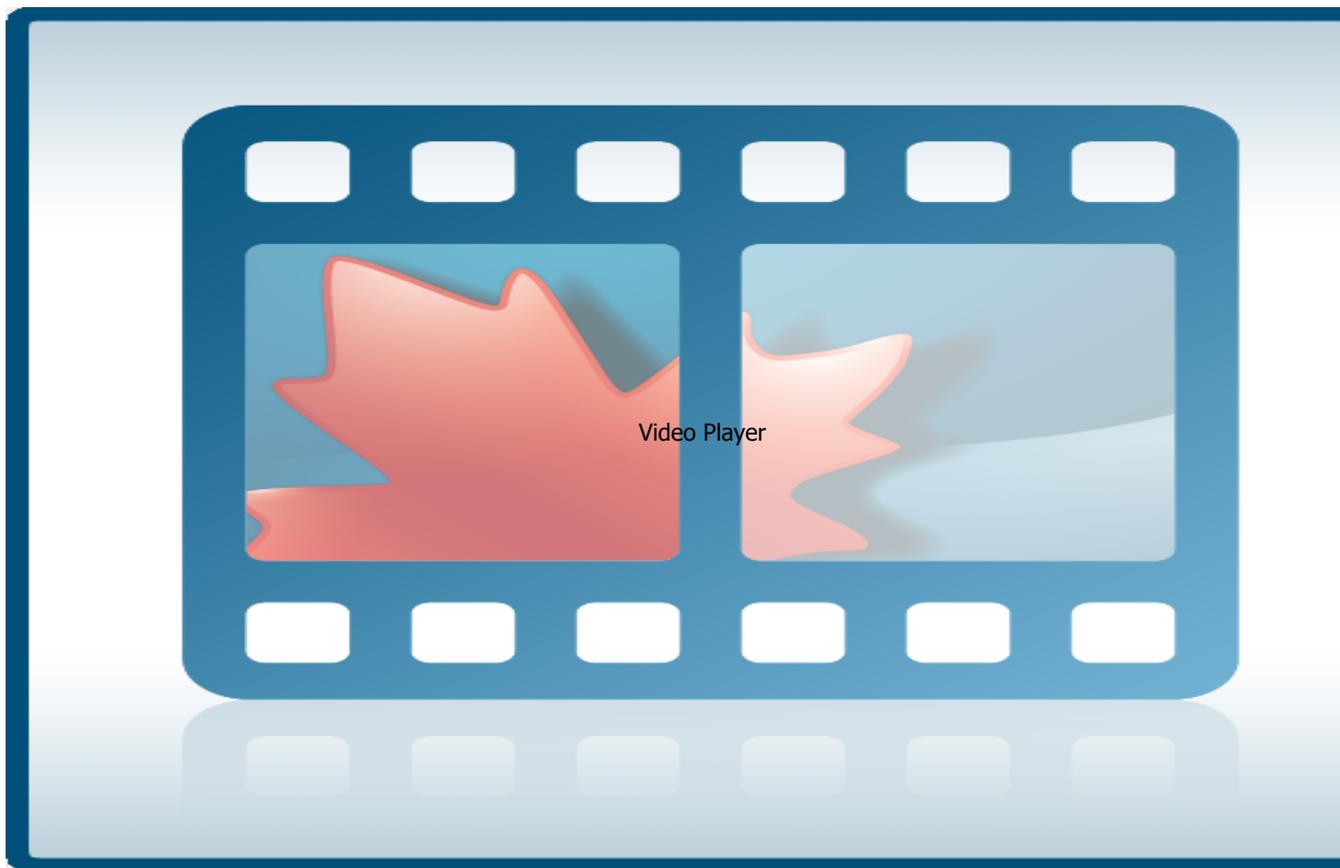
Step 5: Set up the visualization (Inserting the **Visualization** components is optional)

1. Click the **Cylindrical Geometry** component and enter a value around **0.01 m** for the radius.
2. Click the **Spherical Geometry** component and enter a value around **0.05 m** for the radius.
3. Click the **Spring Geometry** component, enter a number around **10** for the number of windings, enter a value around **0.02 m** for **radius1** and enter a value around **0.005 m** for **radius2**.

Step 6: Run the Simulation

1. Click the **Probe** attached to the **Revolute** joint and select **Angle**, **Speed** and **Acceleration** to obtain plots of the angular position, speed and acceleration vs. time.
2. In the **Settings** tab, change the **Simulation duration** to around **2-4 s** to make the plots that are generated easier to read.
3. Run the simulation.

The following video shows the visualization of the simulation.



Video 1: Simulation visualization

Reference:

D. J. Inman. "Engineering Vibration", 3rd Edition. Upper Saddle River, NJ, 2008, Pearson Education, Inc.