Automated derivation of several geometry theorems from a given one (with Maple's help)

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GEOGEBRA DISCOVERY

- Fork version of GeoGebra with expanded automated reasoning features
- Different versions, available off and online, on different operating systems and devices
- Relation, Prove, Discover, ShowProof, LocusEquation, etc
- CAS view as Maple to Clipboard

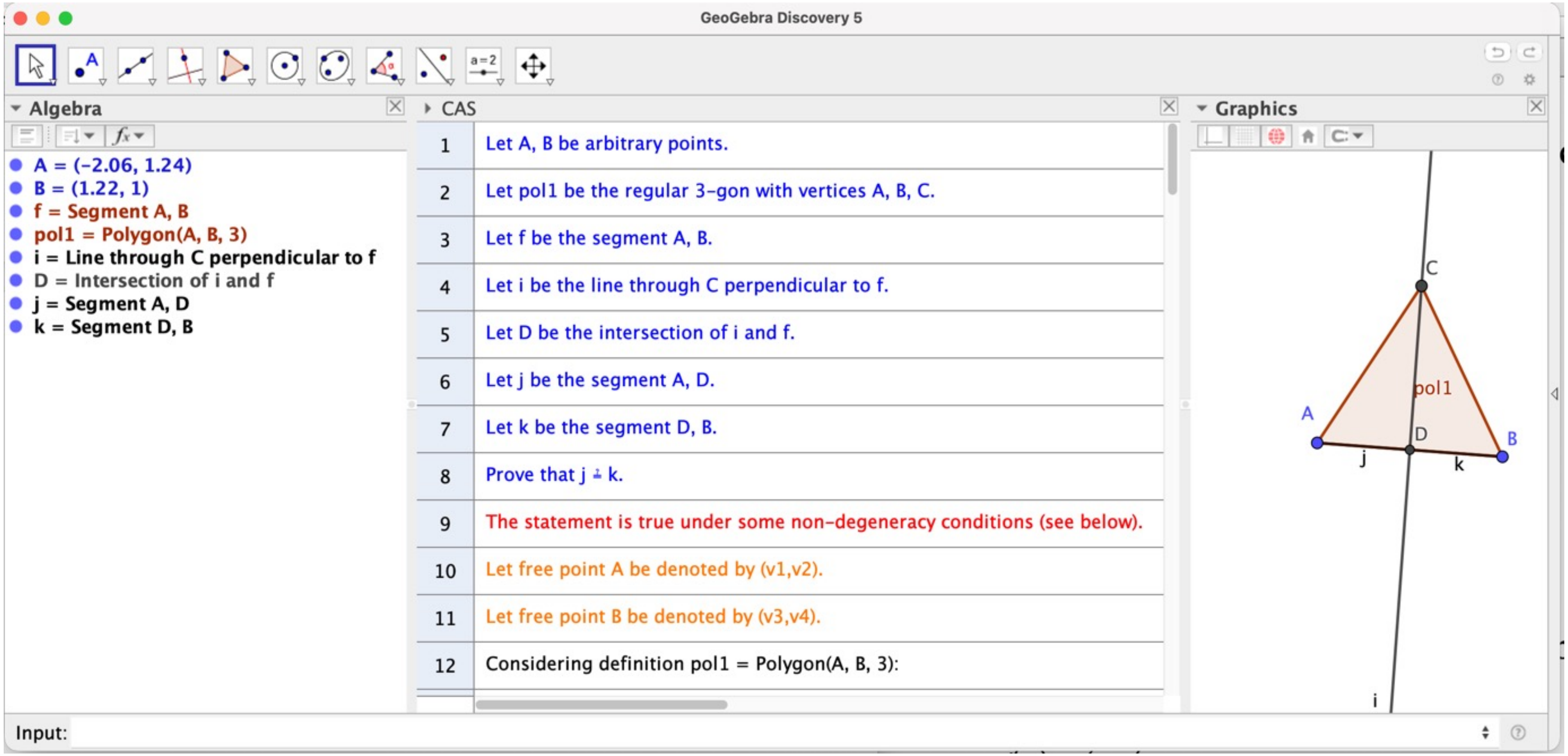
https://github.com/kovzol/geogebra-discovery?tab=readme-ov-file

Kovács, Z., Parisse, B., Recio, T., Vélez, M.P., and Jonathan H. Yu: “The ShowProof command in GeoGebra Discovery: Towards the automated ranking of elementary geometry theorems.” ACM Communications in Computer Algebra, Vol. 58, No. 2, Issue 228, June 2024. pp. 27-30 (published 10 Jan. 2025), **https://doi.org/10.1145/3712023.3712026**

https://www.researchgate.net/publication/382367183_ShowProof_in_GeoGebra_Discovery_Towards_automated_ranking_of_elementary_geometry_theorems

Kov\'acs, Z.; Recio, T.; V\'elez, M.P.: "Automated reasoning tools with GeoGebra: What are they? What are they good for?" In: P. R. Richard, M. P. V\'elez, S. van Vaerenbergh (eds): Mathematics Education in the Age of Artificial Intelligence: How Artificial Intelligence can serve mathematical human learning. Series: Mathematics Education in the Digital Era, Springer, 2022, pp. 23-44. https://doi.org/10.1007/978-3-030-86909-0_2

Given an equilateral triangle, the altitude from a vertex divides the opposite side in two equal segments.



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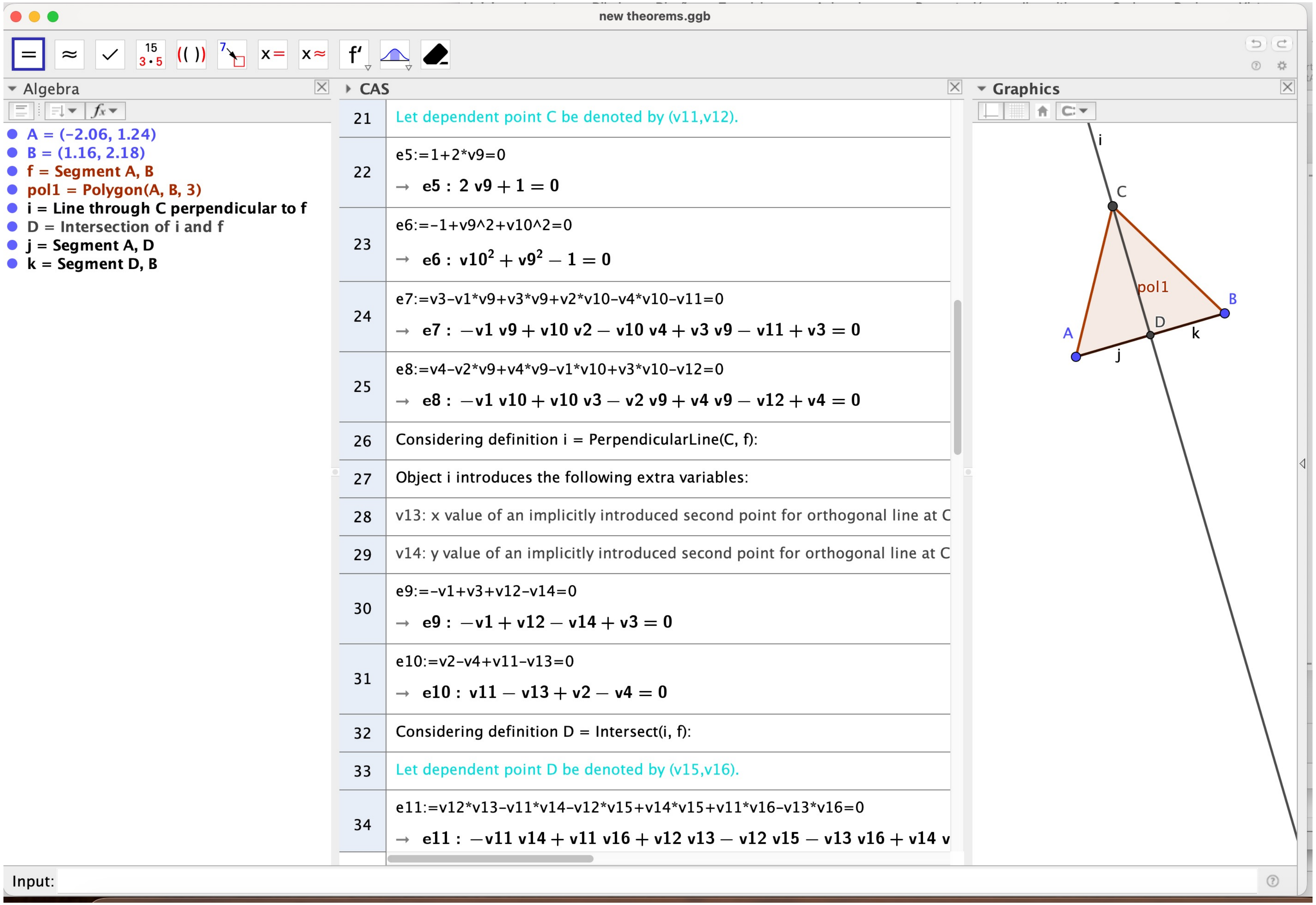
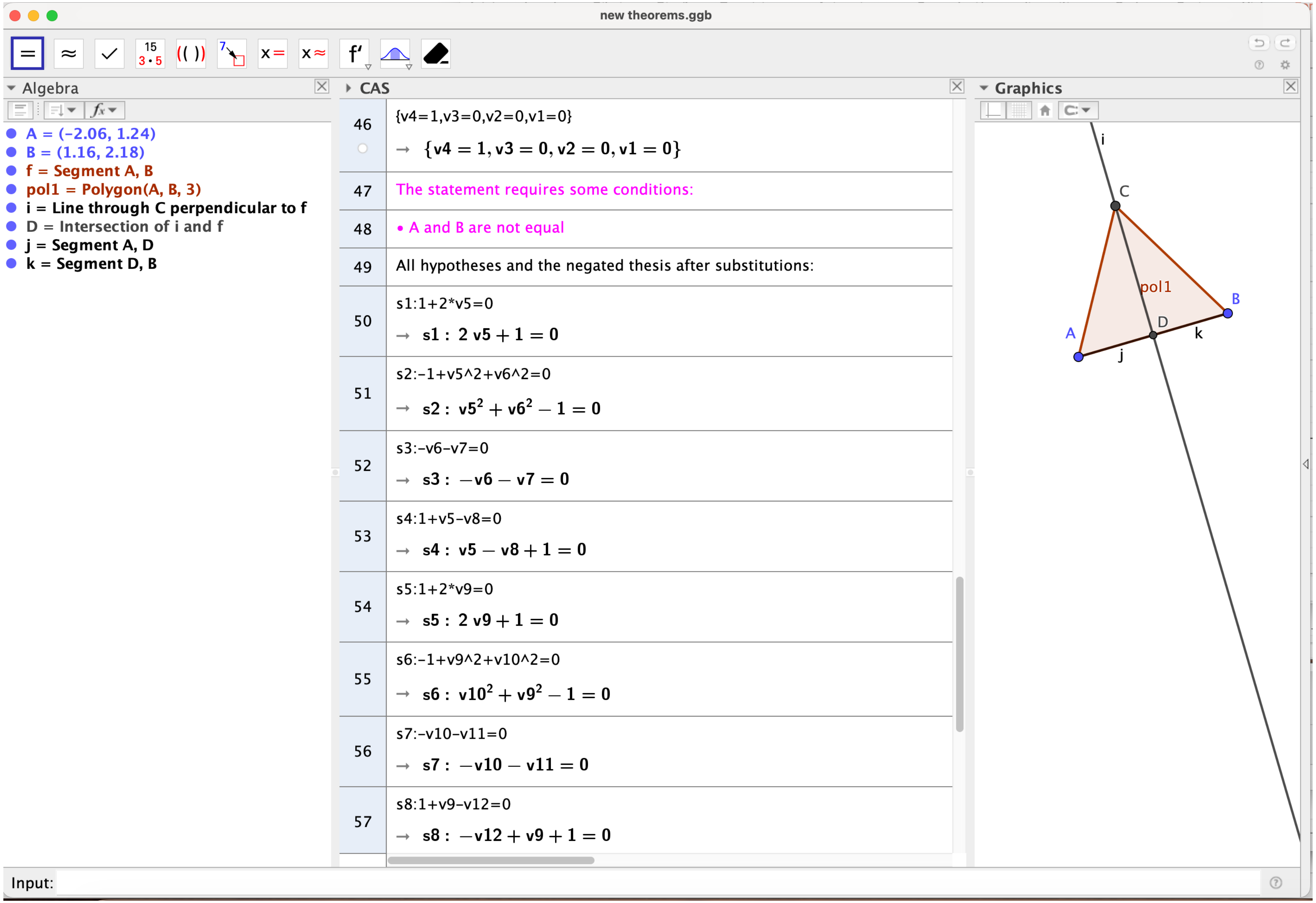

new theorems.ggb
Algebra
A = (-2.06, 1.24)
B = (1.16, 2.18)
f = Segment A, B
pol1 = Polygon(A, B, 3)
i = Line through C perpendicular to f
D = Intersection of i and f
j = Segment A, D
k = Segment D, B
CAS
21 Let dependent point C be denoted by (v11,v12).
22 e5:=1+2*v9=0
→ e5 : 2 v9 + 1 = 0
23 e6:=-1+v9^2+v10^2=0
→ e6 : v10² + v9² − 1 = 0
24 e7:=v3-v1*v9+v3*v9+v2*v10-v4*v10-v11=0
→ e7 : −v1 v9 + v10 v2 − v10 v4 + v3 v9 − v11 + v3 = 0
25 e8:=v4-v2*v9+v4*v9-v1*v10+v3*v10-v12=0
→ e8 : −v1 v10 + v10 v3 − v2 v9 + v4 v9 − v12 + v4 = 0
26 Considering definition i = PerpendicularLine(C, f):
27 Object i introduces the following extra variables:
28 v13: x value of an implicitly introduced second point for orthogonal line at C
29 v14: y value of an implicitly introduced second point for orthogonal line at C
30 e9:=-v1+v3+v12-v14=0
→ e9 : −v1 + v12 − v14 + v3 = 0
31 e10:=v2-v4+v11-v13=0
→ e10 : v11 − v13 + v2 − v4 = 0
32 Considering definition D = Intersect(i, f):
33 Let dependent point D be denoted by (v15,v16).
34 e11:=v12*v13-v11*v14-v12*v15+v14*v15+v11*v16-v13*v16=0
→ e11 : −v11 v14 + v11 v16 + v12 v13 − v12 v15 − v13 v16 + v14 v
Graphics
i
C
pol1
B
A
D
k
j
Input:



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i = Line through C perpendicular to f
D = Intersection of i and f
j = Segment A, D
k = Segment D, B
CAS
46
{v4=1,v3=0,v2=0,v1=0}
→ {v4 = 1, v3 = 0, v2 = 0, v1 = 0}
47
The statement requires some conditions:
48
• A and B are not equal
49
All hypotheses and the negated thesis after substitutions:
50
s1:1+2*v5=0
→ s1 : 2 v5 + 1 = 0
51
s2:-1+v5^2+v6^2=0
→ s2 : v5^2 + v6^2 − 1 = 0
52
s3:-v6-v7=0
→ s3 : −v6 − v7 = 0
53
s4:1+v5-v8=0
→ s4 : v5 − v8 + 1 = 0
54
s5:1+2*v9=0
→ s5 : 2 v9 + 1 = 0
55
s6:-1+v9^2+v10^2=0
→ s6 : v10^2 + v9^2 − 1 = 0
56
s7:-v10-v11=0
→ s7 : −v10 − v11 = 0
57
s8:1+v9-v12=0
→ s8 : −v12 + v9 + 1 = 0
Graphics
i
C
pol1
B
A
D
k
j
Input:

new theorems.ggb

54	s5:1+2*v9=0 → $s5: 2\,v9+1=0$
55	s6:-1+v9^2+v10^2=0 → $s6: v10^2+v9^2-1=0$
56	s7:-v10-v11=0 → $s7: -v10-v11=0$
57	s8:1+v9-v12=0 → $s8: -v12+v9+1=0$
58	s9:v12-v14=0 → $s9: v12-v14=0$
59	s10:-1+v11-v13=0 → $s10: v11-v13-1=0$
60	s11:v12*v13-v11*v14-v12*v15+v14*v15+v11*v16-v13*v16=0 → $s11: -v11\,v14+v11\,v16+v12\,v13-v12\,v15-v13\,v16+v14\,v15=0$
61	s12:-v15=0 → $s12: -v15=0$
62	s13:-1-v17+2*v16*v17=0 → $s13: 2\,v16\,v17-v17-1=0$
63	Now we consider the following equation:
64	s1*(0)+s2*(0)+s3*(0)+s4*(0)+s5*(v11*v17-v13*v17+v15*v17)+s6*(0)+s7*(0)+s8*(-2*v11*v17+2*v13*v17-2*v15*v17)+s9*(-2*v11*v17)+s10*(-2*v16*v17+v17)+s11*(2*v17)+s12*(2*v14*v17-v17)+s13*(-1

Input:

1723x915

60 → $s11: -v11\,v14 + v11\,v16 + v12\,v13 - v12\,v15 - v13\,v16 + v14\,v15 = 0$

61 s12:-v15=0

→ $s12: -v15 = 0$

62 s13:-1-v17+2*v16*v17=0

→ $s13: 2\,v16\,v17 - v17 - 1 = 0$

63 Now we consider the following equation:

64 s1*(0)+s2*(0)+s3*(0)+s4*(0)+s5*(v11*v17-v13*v17+v15*v17)+s6*(0)+s7*(0)+s8*(-2*v11*v17+2*v13*v17-2*v15*v17)+s9*(-2*v11*v17)+s10*(-2*v16*v17+v17)+s11*(2*v17)+s12*(2*v14*v17-v17)+s13*(-1)

→ $1 = 0$

65 **Contradiction! This proves the original statement.**

66 The statement has a difficulty of degree 2.

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Input:

```
s5:=1 + (2 * v9) = 0;
s6:=-1 + v9^(2) + v10^(2) = 0;
s7:=(-v10) - v11 = 0;
s8:=1 + v9 - v12 = 0;
s9:=v12 - v14 = 0;
s10:=-1 + v11 - v13 = 0;
s11:=(v12 * v13) - (v11 * v14) - (v12 * v15) + (v14 * v15) + (v11 * v16) - (v13 * v16) = 0;
s12:=(-v15) = 0;
s13:=-1 - v17 + ((2 * v16) * v17) = 0;


>expand( (s5 * ((v11 * v17) - (v13 * v17) + (v15 * v17))) + (s6 * 0) + (s7 * 0) + (s8 * (((-2 * v11) * v17) + ((2 * v13) * v17) - ((2 * v15) * v17))) + (s9 * ((-2 * v11) * v17)) + (s10 * (((-2 * v16) * v17) + v17)) + (s11 * (2 * v17)) + (s12 * (((2 * v14) * v17) - v17)) + (s13 * (-1)));
                     1 = 0
```

Thesis:(-1+2*v16=0), that is s13:= -1+thesis*v17=0

Therefore, substituting v17 by 1/thesis, yields s13 identically zero. Next, we forget s13 and multiply the resulting expression by (-1+2*v16) to eliminate denominators, bearing in mind that the result will be equal to 1 multiplied by (-1+2*v16) , ie. the thesis itself :

```
> simplify((s5 - 2*s8 - 2*s9)*v11 + (v15 - v13)*s5 + (-2*v15 + 2*v13)*s8 + (-2*v16 + 1)*s10 + s12*(2*v14 - 1) + 2*s11);
```

$$-1 + 2\ v16 = 0$$

(v11 + v15 - v13)*s5 + (-2*v11 - 2*v15 + 2*v13)*s8 - 2*s9*v11 + (-2*v16 + 1)*s10 + 2*s11 + (2*v14 - 1)*s12 - thesis = 0

Derivation 1

Since hypotheses s6 and s7 have 0 as polynomial multiplier in the above identity, we could say that we do NOT need to consider C as the rotation by 120 degrees of the symmetrical of A with respect to B, but just leave x-coordinate of C (v11) free, and place the y-coordinate v12 so that ABC is isosceles (s8: v12= 1+v9= 1/2).

(v11 + v15 - v13)*s5 + (-2*v11 - 2*v15 + 2*v13)*s8 - 2*s9*v11 + (-2*v16 + 1)*s10 + 2*s11 + (2*v14 - 1)*s12 - thesis = 0

In summary, we have derived the fact that the feet of the perpendicular from C to AB is the mid of AB holds also for isosceles triangles at C.

Derivation 2

(v11 + v15 - v13)*s5 + (-2*v11 - 2*v15 + 2*v13)*s8 - 2*s9*v11 + (-2*v16 + 1)*s10 + 2*s11 + (2*v14 - 1)*s12 - thesis = 0;

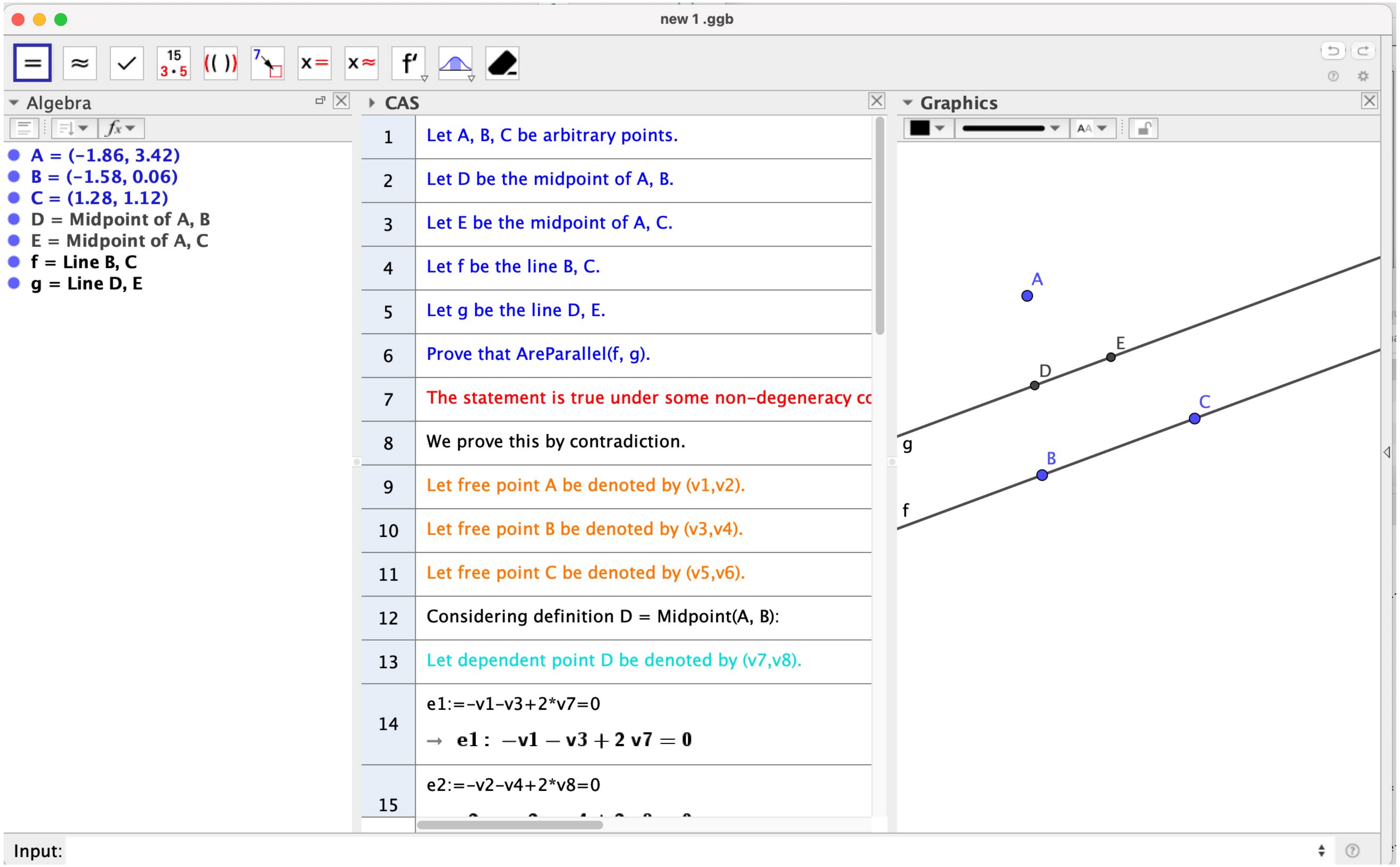
In particular, this implies

(v11 + v15 - v13)*s5 + (-2*v11 - 2*v15 + 2*v13)*s8 - 2*s9*v11 + (-2*v16 + 1)*s10 + 2*s11 - thesis = (1 - 2*v14)*s12;

Thus, we have the following new statement (obvious since s5,s8,s9 imply (1-2*v14)=0)

Hypotheses: {s5, s8, s9, s10, s11, thesis} => (1-2v14)=0 or s12 (v15=0)

Statement: given three points A, B, C, the line joining the midpoints D, E of AB and AC is parallel to line BC



Derivation 3

A(0,0), B(0,1), C(v5,v6), D(v7,v8), E(v9,v10)

s1:=(2 *v7) = 0;s 2:=-1 + (2 * v8) = 0; D midpoint AB

s3:=(-v5) + (2 * v9) = 0; s4:=(-v6) + (2 * v10) = 0; E midpoint AC

Thesis (parallelism BC//DE)=: -v10*v5 + v5*v8 - v6*v7 + v6*v9 + v7 - v9 = 0

Thesis=(-v10 + 1/2)*s1 + v9*s2 + (-v8 + v10)*s3 + (v7 - v9)*s4

Now consider s1, s3,s4, Thesis as hypotheses and s2 as new thesis

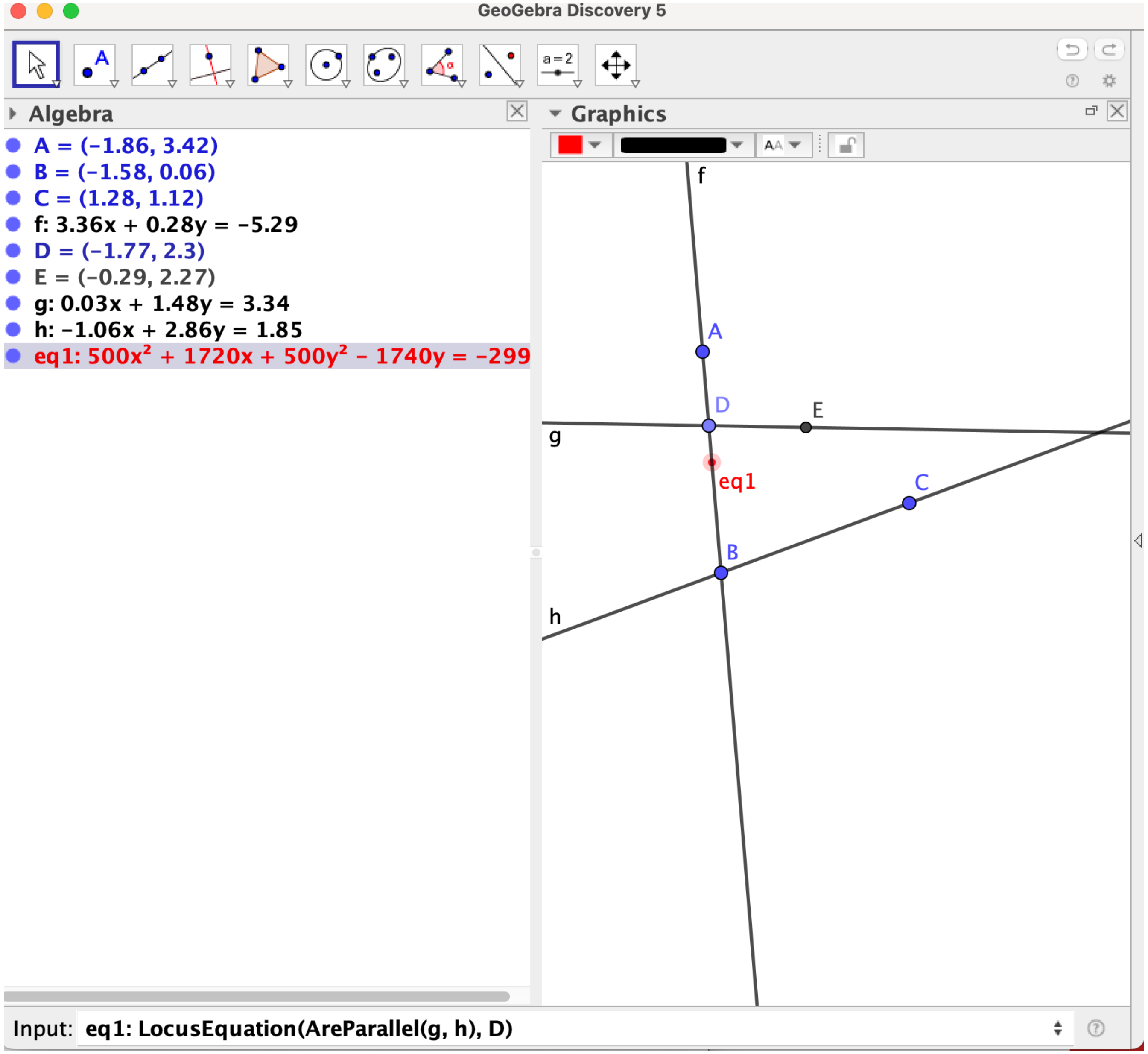
Assume s3 and s4 (ie. E midpoint AC) and s1 (first coordinate of D is 0, namely: D is aligned with A, B (since both have first coordinate= 0); equivalently: first coordinate of the D coincides with the first coordinate of the midpoint of AB.

Then, by s3, C is not aligned with A, B iff v5 is not zero iff v9 is not zero iff E is not aligned with A, B, so we can state a new theorems as follows

If E is the mid point of AC and D is aligned with A, B, and BC is parallel to DE, then, if C is not aligned with A, B (v9 not zero equiv. v5 not zero) D is the midpoint of AB (ie. s2=0, since s1=0 is already true as D is aligned with A, B).

s2=(1/v9)*(thesis (parallelism BC/DE)- ((-v10 + 1/2)*s1 +(-v8 + v10)*s3 + (v7 - v9)*s4))

It is a sort of converse of the midpoint Thales theorem that was the original statement.


GeoGebra Discovery 5
Algebra
A = (−1.86, 3.42)
B = (−1.58, 0.06)
C = (1.28, 1.12)
f: 3.36x + 0.28y = −5.29
D = (−1.77, 2.3)
E = (−0.29, 2.27)
g: 0.03x + 1.48y = 3.34
h: −1.06x + 2.86y = 1.85
eq1: 500x² + 1720x + 500y² − 1740y = −299
Graphics
f
A
D
E
g
eq1
C
B
h
Input: eq1: LocusEquation(AreParallel(g, h), D)

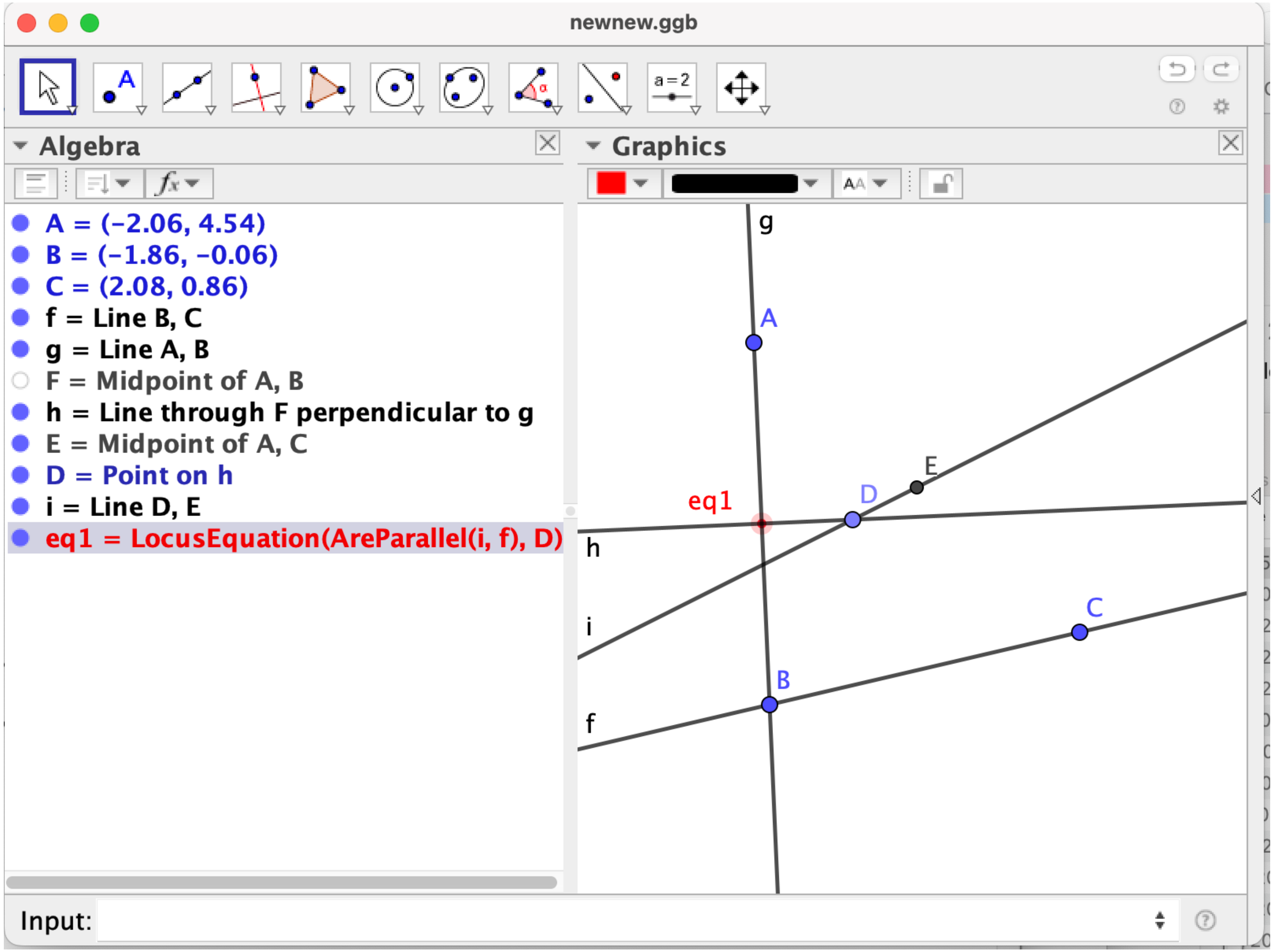
Derivation 4

Another statement, now take s1 as thesis (D has first coordinate as midpoint of AB, or D is aligned with AB)

(-v10 + 1/2)*s1=Thesis (parallelism BC//DE)-(v9*s2 + (-v8 + v10)*s3 + (v7 - v9)*s4)

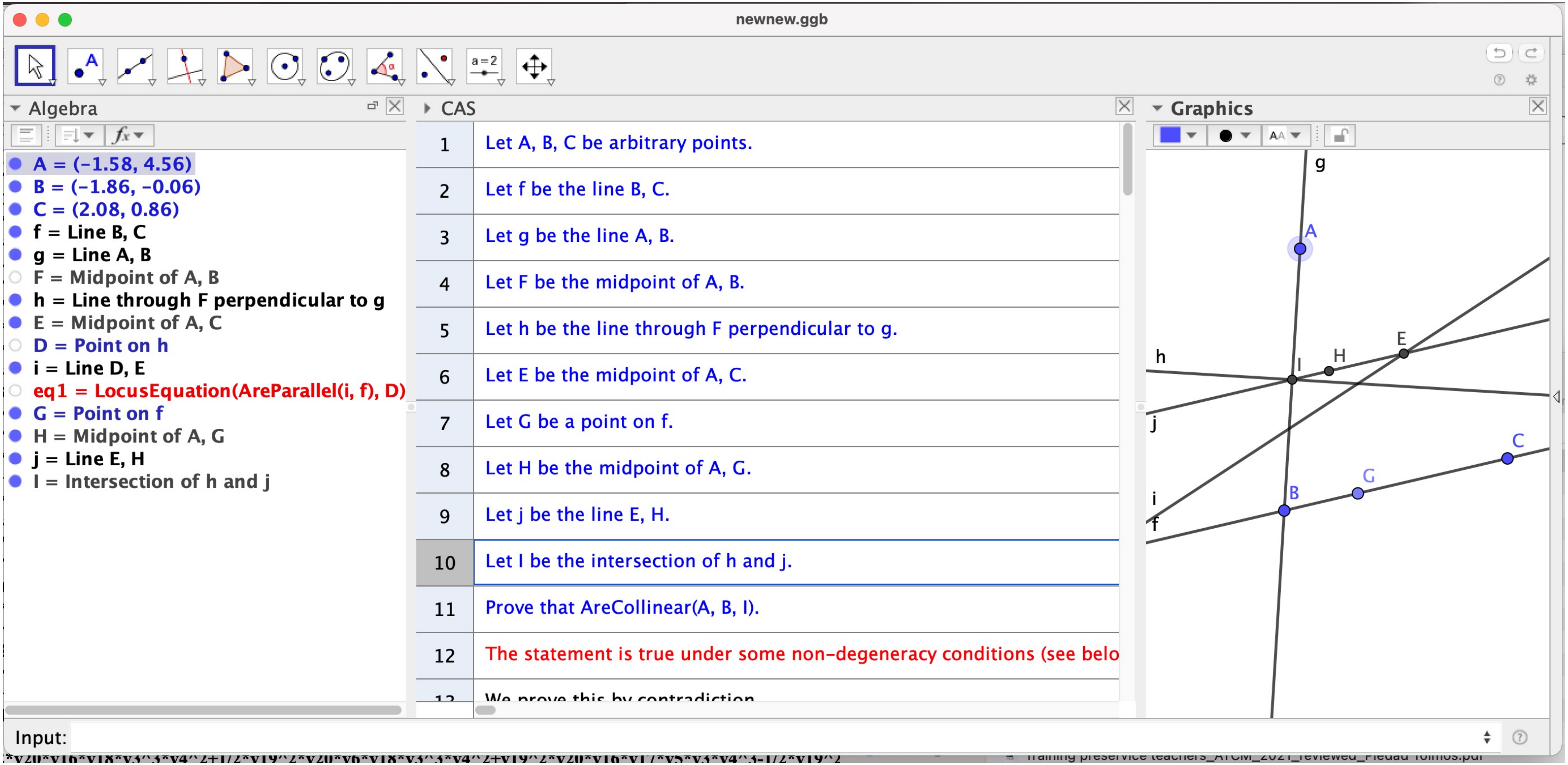
Statement: *if BC is parallel to DE and second coordinate of D is mid of AB, and E is mid AC and second coordinate of E is not mid AB then D is aligned with AB.*

High complexity.


newnew.ggb
Algebra
A = (−2.06, 4.54)
B = (−1.86, −0.06)
C = (2.08, 0.86)
f = Line B, C
g = Line A, B
F = Midpoint of A, B
h = Line through F perpendicular to g
E = Midpoint of A, C
D = Point on h
i = Line D, E
eq1 = LocusEquation(AreParallel(i, f), D)
Graphics
g
A
E
D
eq1
h
i
C
B
f
Input:

We can not find directly the complexity of this new statement with GGDiscovery, since we can not construct the statement without placing D in the line AB. So we can do some turnaround and translate this statement as follows:

We construct a free point G on line BC and a midpoint H of AG. Then the line j= EH will be always parallel to BC. Now we request that H has also second coordinate as the midpoint of AB, so H must be in the line h, perpendicular to AB through its mid point F. So, the hypotheses implies that our point D must be in the intersection of h and j (point I). And the thesis is that then A, B, and I are collinear.


newnew.ggb
Algebra
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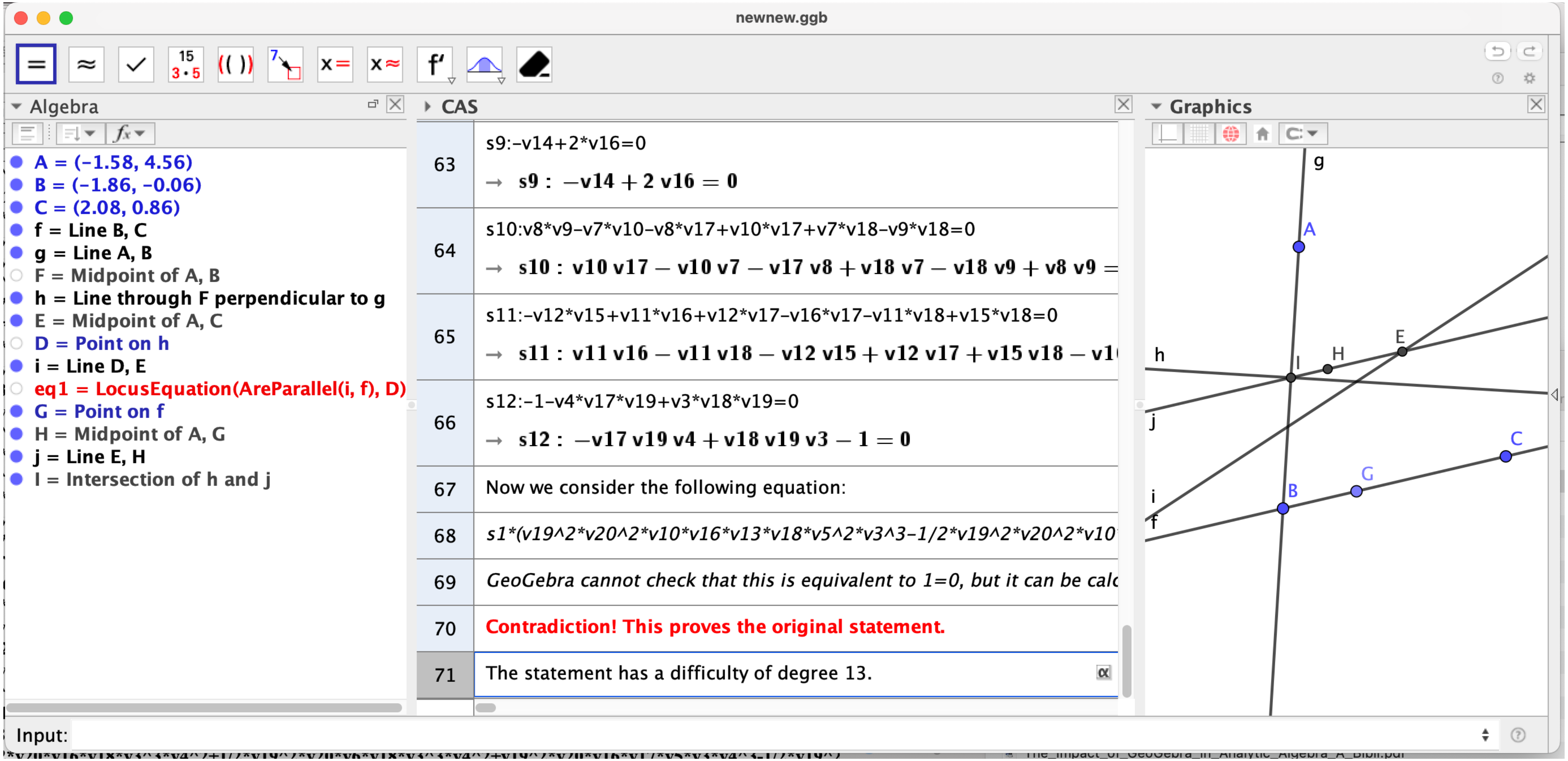
B = (-1.86, -0.06)
C = (2.08, 0.86)
f = Line B, C
g = Line A, B
F = Midpoint of A, B
h = Line through F perpendicular to g
E = Midpoint of A, C
D = Point on h
i = Line D, E
eq1 = LocusEquation(AreParallel(i, f), D)
G = Point on f
H = Midpoint of A, G
j = Line E, H
I = Intersection of h and j
CAS
1 Let A, B, C be arbitrary points.
2 Let f be the line B, C.
3 Let g be the line A, B.
4 Let F be the midpoint of A, B.
5 Let h be the line through F perpendicular to g.
6 Let E be the midpoint of A, C.
7 Let G be a point on f.
8 Let H be the midpoint of A, G.
9 Let j be the line E, H.
10 Let I be the intersection of h and j.
11 Prove that AreCollinear(A, B, I).
12 The statement is true under some non-degeneracy conditions (see belo
Graphics
g
A
h
I
H
E
j
C
G
B
i
f
Input: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F = Midpoint of A, B
h = Line through F perpendicular to g
E = Midpoint of A, C
D = Point on h
i = Line D, E
eq1 = LocusEquation(AreParallel(i, f), D)
G = Point on f
H = Midpoint of A, G
j = Line E, H
I = Intersection of h and j
CAS
63 s9:-v14+2*v16=0
→ s9 : −v14 + 2 v16 = 0
64 s10:v8*v9-v7*v10-v8*v17+v10*v17+v7*v18-v9*v18=0
→ s10 : v10 v17 − v10 v7 − v17 v8 + v18 v7 − v18 v9 + v8 v9 =
65 s11:-v12*v15+v11*v16+v12*v17-v16*v17-v11*v18+v15*v18=0
→ s11 : v11 v16 − v11 v18 − v12 v15 + v12 v17 + v15 v18 − v1
66 s12:-1-v4*v17*v19+v3*v18*v19=0
→ s12 : −v17 v19 v4 + v18 v19 v3 − 1 = 0
67 Now we consider the following equation:
68 s1*(v19^2*v20^2*v10*v16*v13*v18*v5^2*v3^3-1/2*v19^2*v20^2*v10
69 GeoGebra cannot check that this is equivalent to 1=0, but it can be calc
70 Contradiction! This proves the original statement.
71 The statement has a difficulty of degree 13.
Graphics
g
A
h
I
H
E
j
C
G
B
i
f
Input:

Ongoing work.

- Surprising. Hypotheses imply thesis, thesis imply hypotheses?
- Maple is ABSOLUTELY needed to explore and interpret
- Difficult to interpret geometrically
- Some results obtained are simple, some are complex, some are obvious, some are interesting.

Maple file pdf available here
https://www.dropbox.com/scl/fi/yaa6v5t2crgy986ce9hsv/Automated-derivation-1-copia.pdf?rlkey=0u06edr3dmjfvbuwlwejissyp&dl=0

Basic References

Inspired by

"Heuristic strategies for geometry theorem extension based on complex number identity". Xicheng Peng, Jingzhong Zhang, Mao Chen, Sannyuya Liu. Annals of Mathematics and Artificial Intelligence (published 29 July 2025), https://doi.org/10.1007/s10472-025-09988-4

“The ShowProof command in GeoGebra Discovery: Towards the automated ranking of elementary geometry theorems.” Kovács, Z., Parisse, B., Recio, T., Vélez, M.P., and Jonathan H. Yu. ACM Communications in Computer Algebra, Vol. 58, No. 2, Issue 228, June 2024. pp. 27-30 (published 10 Jan. 2025), https://doi.org/10.1145/3712023.3712026

"Detecting truth, just on parts", Z. Kovács, T. Recio, and M. P. Vélez (2019). Revista Matemática Complutense, 32, 451–474, https://doi.org/10.1007/s13163-018-0286-1