

The dual of Philo's shortest line segment problem and its solution with Maple

Maple Conference
Free Virtual Event

Yagub Aliyev

ADA University

November 5-7, 2025

Extrema problems in geometry

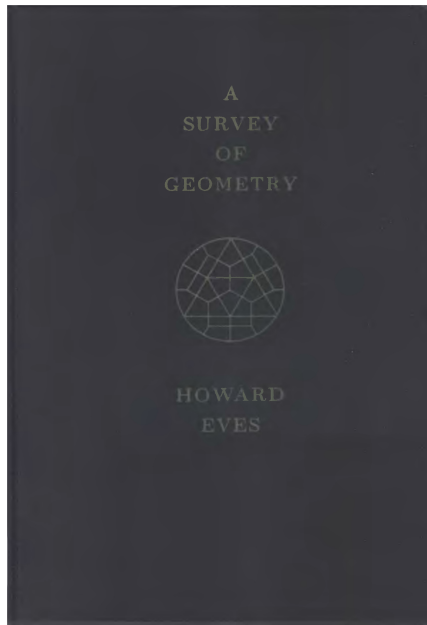
Maximum-minimum problems in geometry played crucial role in the history of development of mathematical methods Courant, Chapter VII; Boltyansky et al; Polya, Chapters VIII-XIX. Many attempts to solve variety of geometry problems for optimization lead to the development of differential and integral calculus and later to the calculus of variations.

R. Courant, H.E. Robbins, Was ist Mathematik?, Springer-Verlag, Berlin, 2010.

Boltyansky V. G., Yaglom I. M., Geometrical problems on maximum and minimum, in Encyclopedia of Elementary Mathematics in Five Volumes. Volume 5. Eds.: P.S. Aleksandrov, A.I. Markushevich, A.I. Khinchin, Moscow, Nauka, 1966. p. 270-348.

G. Polya Mathematics and Plausible Reasoning, Volume 1: Induction and Analogy in Mathematics, Princeton University Press, 1990.

Philo's line



A SURVEY OF GEOMETRY

Volume Two

HOWARD
EVES

*Professor of Mathematics
University of Maine*

ALLYN AND BACON, INC.
BOSTON
1965

Philo's line

XIII. LIMIT OPERATIONS IN GEOMETRY

P on $\hat{O}A$, and S the foot of the perpendicular from P on VT . Then, by elementary geometry,

$$OR|VO = SP|OS$$

or

$$(1) \quad \bar{x}/m = x/y.$$

Again

$$\hat{O}T|VO = VO|OS$$

or

$$(2) \quad \bar{y}/m = n/y.$$

From (1) and (2) we now get

$$\bar{x} = mx/y, \quad \bar{y} = mn/y.$$

13.2.9. THE SMITH-MEHMKE PROJECTIVE INVARIANT. *The ratio of the curvatures of two curves in a plane at a point of contact of the two curves with one another is invariant under projection.*

Let plane p be projected from a center of projection V onto a plane $\hat{p}$, and choose rectangular Cartesian coordinate systems in p and $\hat{p}$ as guaranteed by Theorem 13.2.8. Letting primes denote differentiation with respect to y and $\hat{y}$, one can then find (we leave it to the reader to verify this)

$$\bar{x}'' = m^2 n x'' / \hat{y}^3.$$

Now let $x = f(y)$ and $x = g(\hat{y})$ be two curves in plane p tangent to each other at the point $P: (x_0, y_0)$, and let r denote the ratio of the two curvatures at P . Then, using familiar formulas for curvature from calculus,

$$r = \frac{-f''(y_0)}{(1 + [f'(y_0)]^2)^{3/2}} \frac{(1 + [g'(\hat{y}_0)]^2)^{3/2}}{-g''(\hat{y}_0)} = f''(y_0)/g''(\hat{y}_0),$$

since $f'(y_0) = g'(\hat{y}_0)$. Similarly, the ratio of the curvatures of the two projected curves in plane $\hat{p}$ is

$$\hat{r} = \hat{f}''(\hat{y}_0)/\hat{g}''(\hat{\hat{y}}_0) = [m^2 n f''(y_0)/\hat{y}_0^3]/[m^2 n g''(\hat{y}_0)/\hat{\hat{y}}_0^3] = r,$$

and the theorem is proved.

13.2.10 DEFINITION. The transversal drawn through a given point within a given angle so that the sides of the angle intercept on the transversal a segment of minimum length is known as the *Philo* (or *Philon*) line of the point for the given angle.

13.2 SOME GEOMETRICAL APPLICATIONS OF THE LIMIT CONCEPT

The Philo line is named after Philon of Byzantium, an ancient Greek writer on mechanical devices, who flourished probably in the first or second century B.C., and who devised an interesting reduction of the famous problem of duplicating the cube, in which a special case of the line plays a cardinal role. Because of this connection with the duplication problem, and because of its own inherent attractions, Philo's line has excited interest over the ages. We shall here, with the use of the limit concept, obtain an important geometrical characterization of the line. The method we shall employ will illustrate a valuable technique, known as the *principle of coincidence of equal values*, often employed in locating ordinary maxima and minima.

13.2.11 THEOREM. *Let AB (see Figure 13.2f), with A on CR and B on CS , be the Philo line of a point P for an angle RCS , and let Q be the foot of the perpendicular from C on AB . Then $AP = QB$.*

In Figure 13.2f, let $A'B'$ and $A''B''$ be equally long neighboring segments passing through P . Let U be the foot of the perpendicular from A' on $A'B'$, and V the foot of the perpendicular from B'' on $A'B'$. Take X and Y such that $PX = PA'$ and $PY = PB''$. Then

$$\begin{aligned} A'X &= A''U - XU \\ &= A'U \cot CA'B'' - A'P(1 - \cos A'PA'') \\ &= A'P \sin A'PA'' \cot CA'B'' - A'P(1 - \cos A'PA''). \end{aligned}$$

Similarly,

$$\begin{aligned} YB' &= VB' - VY \\ &= B''V \cot CB'A' - PB''(1 - \cos A'PA'') \\ &= PB'' \sin A'PA'' \cot CB'A' - PB''(1 - \cos A'PA''). \end{aligned}$$

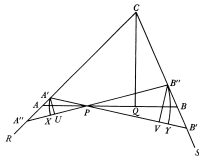


FIG. 13.2f

④

A set of small navigation icons typically found in Beamer presentations, including symbols for back, forward, search, and other slide controls.

В. Ю. Протасов

МАКСИМУМЫ И МИНИМУМЫ В ГЕОМЕТРИИ

лом f' , таким образом, $f'(x)=a$. Для достаточно малых приращений h функция $f(x+h)$ приближённо равна линейной функции $f(x)+ah$, причём чем меньше h , тем это приближение точнее.

54. Через данную точку внутри угла провести отрезок с концами на сторонах угла, имеющий наименьшую длину.

Удивительно, что эта чисто геометрическая задача не имеет столь же ясного геометрического решения. Все более или менее короткие её решения используют производную. Интересно и то, что многие похожие на неё задачи-близнецы, которые, на первый взгляд, даже сложнее её, имеют простые геометрические решения. Например, провести отрезок через данную точку внутри угла, отсекающий от угла треугольник минимальной площади или минимального периметра (задачи 56, 57).

Решение. Обозначим кратчайший отрезок через AB , а данную фиксированную точку внутри угла — через M . Проведём через M другой отрезок $A'B'$ с вершинами на сторонах угла. Пусть δ — угол между $A'B'$ и AB . Функция $f(\delta)=A'B'$ достигает своего минимума в точке $\delta=0$, потому что $f'(0)=0$. Применив теорему синусов к треугольникам MBB' и MAA' , получим

$$MB' = MB \frac{\sin \beta}{\sin(\beta+\delta)}, \quad MA' = MA \frac{\sin \alpha}{\sin(\alpha-\delta)};$$

следовательно,

$$\begin{aligned} f(\delta) - f(0) &= A'B' - AB = MB' + MA' - MB - MA = \\ &= MB \left(\frac{\sin \beta}{\sin(\beta+\delta)} - 1 \right) + MA \left(\frac{\sin \alpha}{\sin(\alpha-\delta)} - 1 \right) = \\ &= -MB \frac{2 \sin \frac{\delta}{2} \cos \left(\beta + \frac{\delta}{2} \right)}{\sin(\beta+\delta)} + MA \frac{2 \sin \frac{\delta}{2} \cos \left(\alpha - \frac{\delta}{2} \right)}{\sin(\alpha-\delta)}. \end{aligned}$$

Итак,

$$\frac{f(\delta) - f(0)}{\delta} = -\frac{2 \sin \frac{\delta}{2}}{\delta} \left(MB \frac{\cos \left(\beta + \frac{\delta}{2} \right)}{\sin(\beta+\delta)} - MA \frac{\cos \left(\alpha - \frac{\delta}{2} \right)}{\sin(\alpha-\delta)} \right).$$

Поскольку $\frac{2 \sin \frac{\delta}{2}}{\delta} \rightarrow 1$ при $\delta \rightarrow 0$, и при этом

$$\frac{\cos \left(\beta + \frac{\delta}{2} \right)}{\sin(\beta+\delta)} \rightarrow \operatorname{ctg} \beta, \quad \frac{\cos \left(\alpha - \frac{\delta}{2} \right)}{\sin(\alpha-\delta)} \rightarrow \operatorname{ctg} \alpha,$$

получаем окончательно

$$f'(0) = -MB \operatorname{ctg} \beta + MA \operatorname{ctg} \alpha.$$

Sir I. Newton's example

pose $x = 1 + z$. Then by Prob. 1. $x = z$. So that for $y = xxy$, may be wrote $y = xy + xzy$. Now since $x = 0$, it is plain, that though the Quantities x and z be not of the same length, yet that they flow alike in respect of y , and that they have equal contemporaneous parts. Why therefore may I not represent by the same Symbols Quantities that agree in their Rate of Flowing; and to determine their contemporaneous Differences, why may not I use $y = xy + xzy$ instead of $y = xxy$?

60. Lastly it appears plainly in what manner the contemporary parts may be found, from an Equation involving flowing Quantities. Thus if $y = \frac{x}{x} + x$ be the Equation, when $x = 2$, then $y = 2\frac{1}{2}$. But when $x = 3$, then $y = 3\frac{1}{2}$. Therefore while x flows from 2 to 3, y will flow from $2\frac{1}{2}$ to $3\frac{1}{2}$. So that the parts described in this time are $3 - 2 = 1$, and $3\frac{1}{2} - 2\frac{1}{2} = 1$.

61. This Foundation being thus laid for what follows, I shall now proceed to more particular Problems.

See Simpons Doctores Application of Fluxions T. 1. p. 143 &c. P R O B. III. To determine the Maxima and Minima of Quantities.

1. When a Quantity is the greatest or the least that it can be, at that moment it neither flows backwards or forwards. For if it flows forwards, or increases, that proves it was less, and will presently be greater than it is. And the contrary if it flows backwards, or decreases. Wherefore find its Fluxion, by Prob. 1. and suppose it to be nothing.

2. EXAMP. 1. If in the Equation $x^3 - ax^2 + axy - y^3 = 0$ the greatest Value of x be required; find the Relation of the Fluxions of x and y , and you will have $3xx^2 - 2axx + axy - 3y^2 + axy = 0$. Then making $x = 0$, there will remain $-3y^2 + axy = 0$, or $3y^2 = ax$. By the help of this you may exterminate either x or y out of the primary Equation, and by the resulting Equation you may determine the other, and then both of them by $-3y^2 + ax = 0$.

3. This Operation is the same, as if you had multiply'd the Terms of the proposed Equation by the number of the Dimensions of the other flowing Quantity y . From whence we may derive the

famous

2.

famous Rule of *Huddeus*, that, in order to obtain the greatest or least Relate Quantity, the Equation must be disposed according to the Dimensions of the Correlate Quantity, and then the Terms are to be multiply'd by any Arithmetical Progression. But since neither this Rule, nor any other that I know yet published, extends to Equations affected with surd Quantities, without a previous Reduction; I shall give the following Example for that purpose.

4. EXAMP. 2. If the greatest Quantity y in the Equation $x^3 - ay^3 + \frac{by}{x} - xx\sqrt{ay + xx} = 0$ be to be determin'd, seek the Fluxions of x and y , and there will arise the Equation $3xx^2 - 2ayy + \frac{by}{x^2} - \frac{ayy + 2xy}{\sqrt{ay + xx}} = 0$. And since by supposition $y = 0$, omit the Terms multiply'd by y , (which, to shorten the labour, might have been done before, in the Operation,) and divide the rest by xx , and there will remain $3x - \frac{2ay + 2xy}{\sqrt{ay + xx}} = 0$. When the Reduction is made, there will arise $4ay + 3xx = 0$, by help of which you may exterminate either of the quantities x or y out of the proposed Equation, and then from the resulting Equation, which will be Cubical, you may extract the Value of the other.

5. From this Problem may be had the Solution of these following.

I. In a given Triangle, or in a Segment of any given Curve, to inscribe the greatest Rectangle.

II. To draw the greatest or the least right Line, which can lie between a given Point, and a Curve given in position. Or, to draw a Perpendicular to a Curve from a given Point.

III. To draw the greatest or the least right Lines, which passing through a given Point, can lie between two others, either right Lines or Curves.

IV. From a given Point within a Parabola, to draw a right Line, which shall cut the Parabola more obliquely than any other. And to do the same in other Curves.

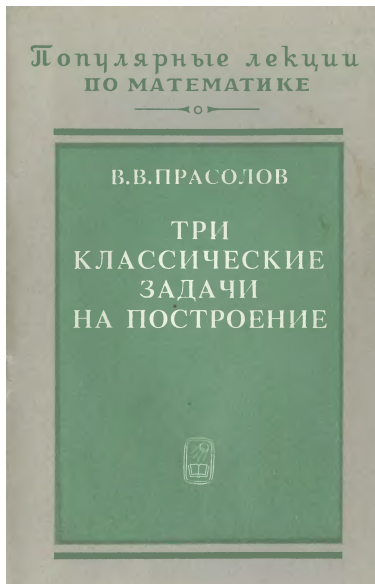
V. To determine the Vertices of Curves, their greatest or least Breadths, the Points in which revolving parts cut each other, &c.

VI. To find the Points in Curves, where they have the greatest or least Curvature.

VII. To find the least Angle in a given Ellipse, in which the Ordinates can cut their Diameters.

VIII.

Duplication of a cube



$xy = r(u+w)$ и $x^2 + 2xu + u^2 = y^2 + 2ry + r^2$, получаем $xy = ab$ и $x(x+b) = y(y+a)$. Следовательно,

$$\frac{a}{x} = \frac{y}{b} = \frac{a+y}{x+b} \quad \text{и} \quad \frac{x}{y} = \frac{v+a}{x+b} = \frac{a}{b} = \frac{y}{b}.$$

Никомед в качестве искоемых отрезков указал отрезки CK и MA ; он, по-видимому, не заметил, что $MA = FN$.

Своим решением Никомед очень гордился и считал, что оно гораздо лучше построения Эратосфена, которое он высмеивал как непрактичное и негеометрическое.

Легко проверить, что конхоида задается уравнением четвертой степени. В самом деле, из рис. 13 видно, что $\sqrt{x^2 + y^2} : y = r : (y-a)$, т. е. $(x^2 + y^2)(y-a)^2 - r^2 y^2 = 0$.

Папп Александрийский показал, что «вставление» отрезка между прямыми можно свести к нахождению точки пересечения окружности и гиперболы. Но наш рассказ о способе «вставок» уже слишком затянулся, поэтому мы отложим обсуждение этого до разговора о трисекции угла (см. с. 37). Тем более что и сам Папп занимался сведением способа «вставок» к пересечению гиперболы и окружности именно в связи с трисекцией угла.

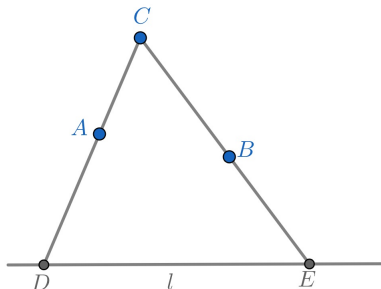
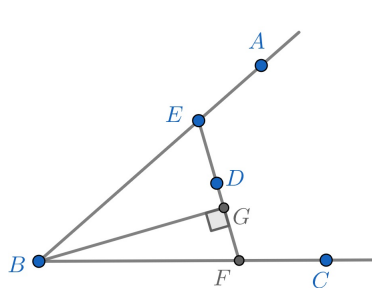
Решения Аполлония, Филона Византийского и Герона

Три математика древности, Аполлоний (III в. до н. э.), Филон Византийский (III в. до н. э.) и Герон (I в. н. э.) в разное время предложили фактически одно и то же решение задачи удвоения куба. Но они не указали, с помощью каких инструментов можно было бы осуществить такое построение.

Рассмотрим прямоугольник $ABDC$, где AB и AC — данные отрезки. Пусть E — точка пересечения диагоналей этого прямоугольника. Для решения задачи удвоения куба достаточно выполнить любое из следующих эквивалентных построений (рис. 14):

- 1) провести окружность с центром E так, чтобы точка D лежала на отрезке, соединяющем точки пересечения этой окружности с лучами AB и AC (Аполлоний);
- 2) провести через точку D прямую так, чтобы описанная окружность прямоугольника $ABDC$ и прямые AB и AC высекали на ней равные отрезки GH и DF (Филон);

Dual Problem



Aliyev Y.N., Construction of Tangents of Conchoid of Nicomedes and Limacon of Pascal, and Solution of Extremal Chord Problem in Ellipse Using Ophiuride Curve, Tr. J. Nature Sci., Vol. 2 No. 2 (2013) 1-6.

Duality of lines and points

	Philo's line problem	Dual problem
Fixed point(s)	1 (D)	2 (A and B)
Fixed line(s)	2 (BA and BC)	1 (I)
Moving point(s)	2 (E and F)	1 (C)
Moving line(s)	1 (EF)	2 (CD and CE)

Table: The number of elements in Philo's line problem and its dual.



ISSN: 2149-6366

Yıl/Year: 2013

Cilt/Volume: 2

Sayı/Number: 2

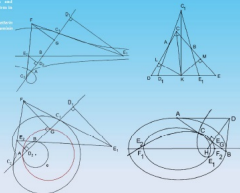
TÜRK DOĞA VE FEN DERGİSİ TURKISH JOURNAL OF NATURE AND SCIENCE

Bilgi Üniversitesi Fen Bilimleri Enstitüsü
tarafından yayımlanmaktadır

Published by Institute of Science of Bilgi
University

Construction of Tangents of Conchoid of Nicomedes and
Limaçon of Pascal, and Solution of Extremal Chord Problem in
Ellipse Using Optimalistic Curve

Nikomedes Konoid ve Paskalın Salıyazıcı Eğrilerine Teğetlerin
Çizilmesi ve Elipse Üz Uzun veya En Kısa Kirişin Bulunması
Probleminin Ofiürid Eğrisi ile Çözülmesi



<http://www.bilgi.edu.tr/tr/dergiler/turk-doga-ve-fen-dergisi.aspx>



Tr. Doğa ve Fen Derg. – Tr. J. Nature Sci. 2013 Vol. 2 No. 2

Türk Doğa ve Fen Dergisi
Turkish Journal of Nature and Science

<http://www.bilgi.edu.tr/dergiler/turk-doga-ve-fen-dergisi.aspx>



Construction of Tangents of Conchoid of Nicomedes and Limaçon of Pascal, and Solution of Extremal Chord Problem in Ellipse Using Optiuride Curve

Yagub N. Aliyev*

Abstract

A new method to construct a tangent to the conchoid of Nicomedes or limaçon of Pascal curves is discussed. Some interesting properties of the conchoid curve (which is a special case of limaçon of Pascal) are investigated. The following problem is studied: "Given a line l and two points A and B on one side of l , find point C such that the sum of lengths of segments CD and CE is minimal, where D and E are intersections of line l with lines CA and CB , respectively." This problem is dual to the classic problem to find shortest segment inscribed to a given angle and passing through a given point. Part of this problem was solved and the remaining part is left as an open question. The problem to find ellipse's tangent or shortest chord passing through a given point, is also considered. For the solution the curve named optiuride is used.

Keywords: Construction of tangents, Conchoid of Nicomedes, Limaçon of Pascal, extremal chords, curves, optiuride, ellipse, conchoid

Nikomedes Konoid ve Paskalın Salıyazıcı Eğrilerine Teğetlerin Çizilmesi ve Elipse Üz Uzun veya En Kısa Kirişin Bulunması Probleminin Ofiürid Eğrisi ile Çözülmesi

Özet

Nikomedes'in Konoid ve Paskal'ın Salıyazıcı eğrilerine teğetlerin çizilmesi için yeni bir yöntem geliştirilmiştir. Konoid eğrisinin (Paskal eğrisinin özel durumu) bazı ilginç özellikleri araştırılmıştır. Tükr eden l problemi incelenmiştir: "Verilen doğru l ki, bir doğru ve onun bir tarafında iki A ve B noktaları veriliyor. Öyle bir C noktası bulunur ki, CD ve CE parçalarının toplamı minimum olur, burada D ve E noktaları doğrudan CA ve CB ile kesişme noktalarıdır." Bu problem klasik bir problem olan açısı içinde verilen açıya geçen ve l noktasından geçen kesilen kısımları en kısa olan doğru parçasını minimum uzunluğunda bulmak meselesine benzer. Bu problemi bir köşen çizilmez köşen köşen ise açık problem olarak bırakılır. Elipse teğetini bulmak noktası geçen en kısa veya en uzun kiriş bulma problemi de araştırılır. Çözüm için Ofiürid eği kullanılır.

Anahtar Kelimeler: Teğetlerin çizilmesi, Nikomedes Konoidi, Paskal Salıyazıcı, en uzun ve en kısa kiriş, eği, ellipse, konoid

1. Introduction

The curve "conchoid" (greek "shell-like") means "looking like a shell" was invented in ancient Greece by mathematician Nicomedes in an attempt to solve "the duplication of a cube" problem. It could also be used to solve the problem of bisecting the angle (1). These two problems and the problem of quadrature of a circle form the famous trio of ancient geometry problems which can not be

solved using only unmarked ruler and compass. Nicomedes who lived 200 BC was one of the many mathematicians who tried to solve them with the help of various instruments and curves. French mathematician E. Pascal (father of famous scientist B. Pascal) applied the construction of Nicomedes to a circle and a point on its circumference. The resulting curve is called limaçon of Pascal (Pascal "limb" means "shell").

In the present paper some interesting properties of the conchoid and limaçon curves are discussed. A new method to construct the tangent line for these two curves is described. This method is intertwined with the known method for constructing the normal (2), (3). A particular case of conchoid is also studied.

*Department of Mathematics and Informatics, Faculty of Pedagogics, Yagub N. Aliyev University, Baku, Azerbaijan

*Corresponding author, email: yagub@bilgi.edu.tr

Received 19 December 2013

Accepted 26 December 2013

PRÉCIS D'ANALYSE FONCTIONNELLE

□ Soit $f(A) = B$ et soit $\{y_n\}$ une suite de points de B . Considérons la suite $\{x_n\}$ formée d'un antécédent au plus de chaque point $y_n = f(x_n)$. L'ensemble A étant compact, la suite $\{x_n\}$ possède une valeur d'adhérence x_0 . La fonction f étant continue sur A et en particulier en x_0 , la valeur $f(x_0) = y_0 \in B$ est une valeur d'adhérence de la suite $\{y_n\}$. ■

En particulier, si $Y = \mathbb{R}$, c'est-à-dire que f est une fonctionnelle définie sur un ensemble compact A d'un espace métrique, alors $f(A)$ est un ensemble borné fermé de la droite numérique et par suite contient sa borne inférieure α et sa borne supérieure β , i.e. f est bornée sur A et atteint ses bornes inférieure et supérieure sur A .

Remarque. Signalons que si une fonction continue $f(x)$ est définie sur un ensemble M non compact, elle n'atteint pas nécessairement ses bornes supérieure et inférieure sur cet ensemble.

Considérons par exemple dans $C[0, 1]$ l'ensemble de toutes les fonctions $x(t)$ telles que $x(0) = 0$, $x(1) = 1$ et $\max_t |x(t)| \leq 1$.

La fonctionnelle $f(x) = \int_0^1 x^2(t) dt$ est continue sur M mais n'atteint pas sa borne inférieure sur M .

En effet, si $x(t) = t^n$, alors $f(x) = 1/(2n+1)$. Donc $\inf f(x) = 0$. Or il est évident que pour toute courbe continue $x = x(t)$ joignant les points $(0; 0)$ et $(1; 1)$ on a $f(x) > 0$ (de là il s'ensuit en particulier que l'ensemble de courbes envisagé n'est pas compact, bien qu'il soit fermé et borné dans $C[0, 1]$).

Donc avant de se prévaloir du théorème 2, il faut s'assurer de la compacité de l'ensemble sur lequel est définie la fonctionnelle continue. Si l'on conjecture qu'une fonctionnelle continue atteint nécessairement ses bornes supérieure et inférieure sur un ensemble non compact, on risque d'obtenir des résultats fallacieux comme le prouve l'exemple envisagé.

Comme autre exemple à l'appui on se propose de produire une démonstration fautive du cinquième postulat d'Euclide. On sait que le cinquième postulat d'Euclide équivaut à l'hypothèse que la somme des angles d'un triangle au moins est égale à π . On peut démontrer de façon rigoureuse que la somme des angles d'un triangle ne peut être supérieure à π . Montrons que la somme des angles d'un triangle est égale à π . Soit α la borne inférieure de la somme des angles de ce triangle et supposons qu'il existe un triangle ABC (fig. 1) dont la somme des angles atteint sa valeur maximale α . Relions le sommet C à un point D intérieur à AB . Le segment CD

partage le triangle ABC en deux triangles ADC et DCB dont la somme des angles de chacun est $\leq \alpha$. D'autre part, la somme des angles des deux triangles est égale à $\alpha + \pi$. Donc $\alpha + \pi \leq 2\alpha$. Or $\alpha \leq \pi$, donc $\alpha = \pi$. Il existe par conséquent un triangle dont la somme des angles est égale à π , ce qui prouve le cinquième postulat d'Euclide.

La faillie de cette démonstration est l'hypothèse de l'existence d'un triangle dont la somme des angles atteint sa borne supérieure (ce qui, on le voit, est équivalent au cinquième postulat d'Euclide). En géométrie de Lobatchevski la différence entre π et la somme des angles d'un triangle est proportionnelle à l'aire de ce dernier et si cette différence tend vers 0, le triangle se réduit à un point.

Le théorème 2 se généralise au cas des fonctionnelles semi-continues. On dit qu'une fonctionnelle $f(x)$ est *semi-continue inférieurement* (resp. *supérieurement*) si la condition $x_n \rightarrow x$ entraîne $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$ (resp. $f(x) \geq \limsup_{n \rightarrow \infty} f(x_n)$).

On a le théorème suivant pour de telles fonctionnelles.

Théorème 3. Une fonctionnelle $f(x)$ semi-continue inférieurement (resp. supérieurement) et définie sur un ensemble compact est minorée (resp. majorée) sur cet ensemble et y atteint sa borne inférieure (resp. supérieure).

Ce théorème est largement utilisé en calcul des variations, puisque les plus importantes classes de fonctionnelles étudiées sont des classes de fonctionnelles semi-continues.

3. Critère de compacité d'ensembles dans un espace métrique. Donnons un critère général de compacité d'un ensemble d'un espace métrique. Introduisons à cet effet la définition suivante: un ensemble N d'un espace métrique X s'appelle *e-réseau* pour un ensemble M de X si pour tout point x de M il existe un point x_n de N tel que $\rho(x, x_n) < \varepsilon$ (M peut en particulier être confondu avec l'espace X tout entier).

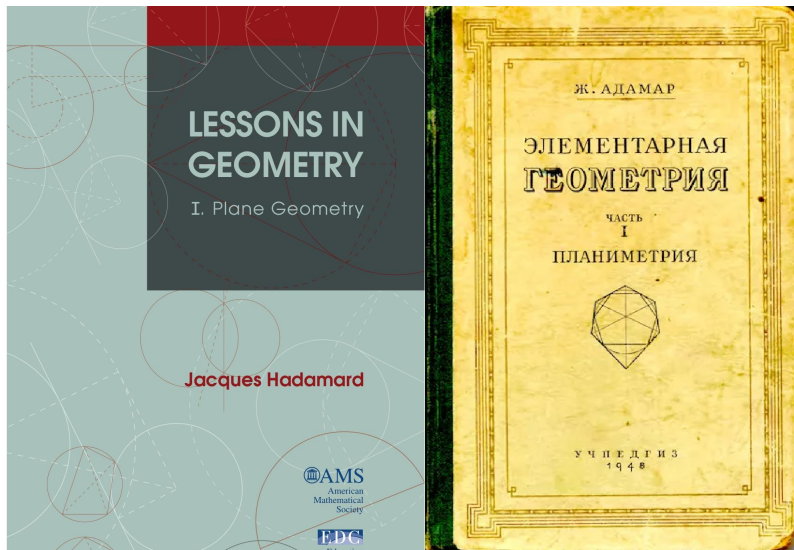
On dit qu'un ensemble M de X est *précompact* si pour tout $\varepsilon > 0$ il existe un *e-réseau* fini pour M . Il est immédiat de s'assurer qu'un ensemble précompact est borné.

Théorème 4 (Hausdorff). Pour qu'un ensemble M de X soit relativement compact, il faut, et si X est complet, il suffit que M soit précompact.

□ *Nécessité.* Supposons que M est relativement compact. Soit x_1 un point de M . Si $\rho(x_1, x_2) < \varepsilon$ pour tout $x \in M$, on a déjà construit un *e-réseau* fini. Sinon il existe un point x_2 de M tel que $\rho(x_1, x_2) \geq \varepsilon$. Si pour tout point x de M soit $\rho(x, x_1) < \varepsilon$, soit $\rho(x, x_2) < \varepsilon$, on a déjà construit un *e-réseau* fini. Sinon il existe un point x_3 de M tel que $\rho(x_1, x_3) \geq \varepsilon$, $\rho(x_2, x_3) \geq \varepsilon$.



Hadamard's geometry book

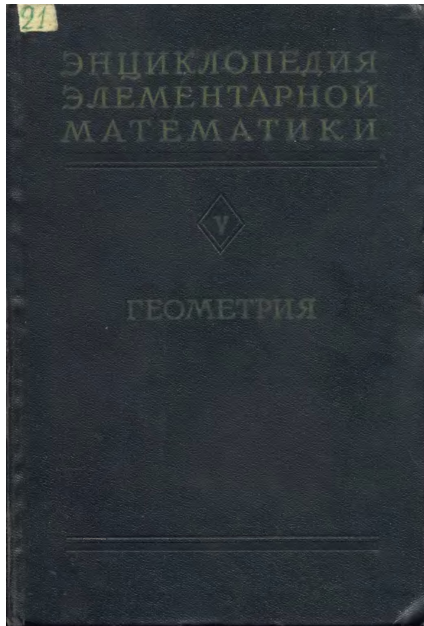


Geometric extrema problems from Hadamard's book

Exercise 366. In a given triangle, inscribe a triangle such that the sum of the squares of its sides is minimal. (Assuming that this minimum exists, show that it can only be the triangle PQR of the preceding exercise.)

Exercise 418b. Among all polygons with the same number of sides, and the same perimeter, the largest is the regular polygon. (Assuming that a polygon of maximum area exists, use the preceding exercises and Exercise 331 to show that this polygon must be regular.)

Analysis and the existence of the extremum



анализом) заключалась в получении необходимых условий, которым должна удовлетворять искомая максимальная (минимальная) фигура, а также в отыскании той фигуры, которая этим необходимым условиям удовлетворяет. Так, условие равенства нулю производной (являющееся необходимым для того, чтобы функция достигала во внутренней точке максимума или минимума) позволило в задачах на стр. 276—278 выделить единственную фигуру, удовлетворяющую этому условию, т. е. единственную фигуру, которая может обладать требуемым свойством минимальности или максимальной. В задаче Торричелли анализ был более сложным, но и в этом случае нам удалось показать, что существует только одна точка, которая может обладать требуемым свойством минимальности.

Однако одним анализом решение рассмотренных задач не ограничивалось. В каждом случае была еще и вторая часть решения, которую мы условимся называть доказательством существования. В этой части решения мы устанавливаем, что фигура, обладающая требуемым свойством максимальной или минимальности, непременно должна существовать. Наконец, после проведения обеих указанных частей решения мы говорили, что, поскольку имеется лишь одна фигура, которая может обладать требуемым свойством минимальности или максимальной, а с другой стороны, фигура, обладающая этим свойством, непременно существует, то, следовательно, найденная фигура действительно обладает требуемым свойством. Этим решение задачи на максимум или минимум и завершалось.

Итак, целью анализа является нахождение фигуры, которая может обладать требуемым свойством максимальной или минимальности¹⁾. Доказательство существования устанавливает, что фигура, обладающая требуемым свойством, непременно существует. Из этих двух частей (анализ и доказательство существования) состояло решение каждой из рассмотренных выше задач. По той же схеме будет построено и решение последующих задач.

Остановимся на вопросе о роли каждого из двух отмеченных этапов решения задачи на отыскание наибольших и наименьших значений. В большинстве случаев основные трудности при решении задачи представляет анализ; доказательство существования чаще всего вытекает естественным образом из хорошо известных общих теорем (теорема Больцано—Вейерштрасса, известная из курсов

¹⁾ Разумеется, может случиться, что существует не одна, а несколько фигур, обладающих требуемым свойством минимальности (или максимальной); в таком случае задача анализа заключается в выделении всех этих фигур.

Indirect and direct proofs

Grundlehren der mathematischen Wissenschaften 360
A Series of Comprehensive Studies in Mathematics

László Fejes Tóth
Gábor Fejes Tóth
Włodzimierz Kuperberg

Lagerungen

Arrangements in the Plane,
on the Sphere, and in Space

the inequality $P \geq \tilde{F}$ the area of P can be replaced by the area of the intersection PK .

Let us now compare this direct proof with the indirect one given before. Generally, the proof of some extremum property can be regarded as direct if it shows directly, without calling on some infinite process, that the corresponding extreme configuration is better than all the other ones in comparison. In this sense we indeed have to rate the first proof as indirect: first the existence of a best polygon is established, and afterwards it is shown that every irregular polygon can be altered so as to get a better one.

Disregarding aesthetic and didactic points of view, the indirect method seems to be more natural and perhaps also more practical. If the primary goal is to find a still unknown extreme figure, one puts the question of existence aside and attacks the following problem: when and how can a figure be improved? However, the indirect method sketched here is neither completely elementary nor purely geometric, since by applying the theorem of WEIERSTRASS it makes use of the elements of analysis.

Consider now the direct proof given above, which by its directness alone seems to be more satisfactory and convincing. The question of existence is not even addressed at first, but remains open and is answered automatically in the end. Moreover, in the above direct proof we use tools from elementary geometry only, while with an indirect proof this is in principle impossible. However, since a direct proof often requires greater skill, such proofs usually emerge only after a "less elegant" solution of the problem has already been found.

To close, we mention the inequality

$$\frac{R}{r} \geq \sec \frac{\pi}{n} \quad (1.3.4)$$

that holds for the inradius r and circumradius R of an arbitrary convex n -gon. This follows directly from the inequality (1.3.1). Furthermore, using the properties of affinity, r^2 and R^2 can be replaced in (1.3.1) by $\frac{e}{e}$ and $\frac{E}{E}$, where e and E denote an ellipse contained in, and containing the n -gon, respectively. Hence, we have the somewhat more general inequality:

$$\frac{E}{e} \geq \sec^2 \frac{\pi}{n}. \quad (1.3.5)$$

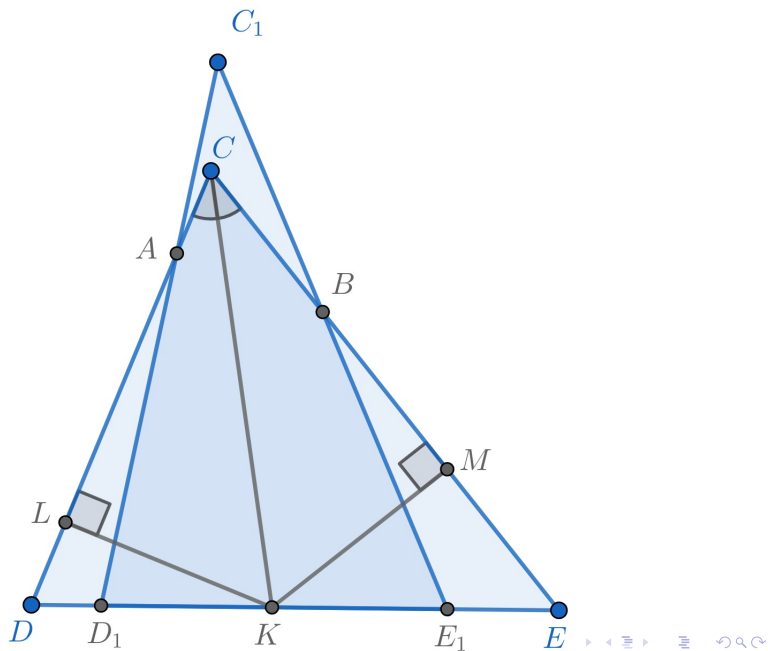
1.4 The Isoperimetric Problem

Among the isoperimetric domains, that is, among the domains of the same perimeter, which one has the greatest area? The solution of this classic, so-called isoperimetric, problem is the circle. In other words: If L denotes the perimeter of a domain of area F in the plane, then

$$L^2 - 4\pi F \geq 0, \quad (1.4.1)$$

and equality holds for the circle only.

Main result



Main result

Theorem

Let CDE be a triangle with acute angles at vertices D and E . Let CK be the angle bisector of triangle CDE . Drop perpendiculars KL and KM to sides CD and CE , respectively. Take points A and B on sides CD and CE , respectively, such that $|AC| = |LD|$, $|BC| = |ME|$. Then for any point C_1 different from C , such that rays C_1A and C_1B intersect line DE at points D_1 and E_1 , respectively, inequality $|C_1D_1| + |C_1E_1| > |CD| + |CE|$ holds true.

Problem

Given $A(a, b)$, $B(c, d)$, with $a < c$ and $b \geq d > 0$ find the minimum of $\max(|C_1D_1|, |C_1E_1|)$, where $C_1(x, y)$ is a point such that $y > b$ and rays C_1A and C_1B intersect line $y = 0$ at points D_1 and E_1 , respectively.

Proof and Solution

Without loss of generality we can assume that $a < c$ and $b \geq d > 0$. We obtain that

$$\begin{aligned} |C_1 D_1| + |C_1 E_1| &= \sqrt{(x - x_1)^2 + y^2} + \sqrt{(x - x_2)^2 + y^2} \\ &= y \left(\sqrt{\left(\frac{x - a}{y - b}\right)^2 + 1} + \sqrt{\left(\frac{x - c}{y - d}\right)^2 + 1} \right) =: f(x, y). \end{aligned}$$

Graph of $f(x, y)$

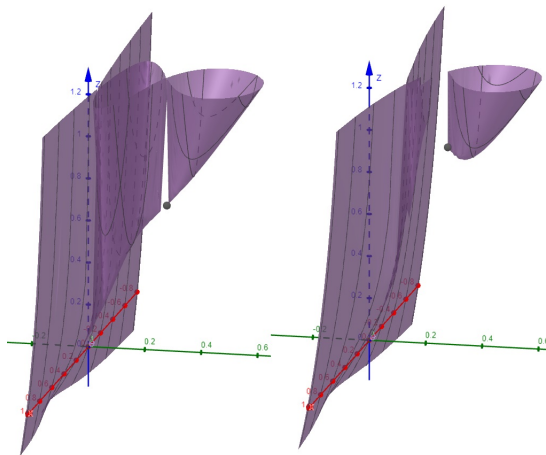
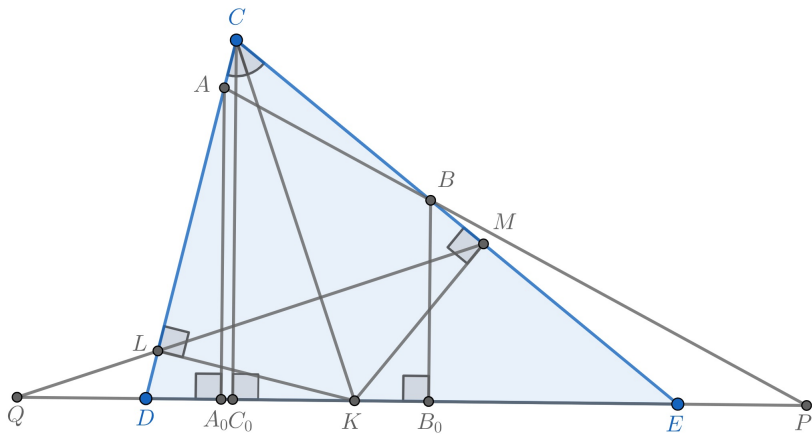


Figure: Graph of f in cases $d - b + \frac{d(c-a)}{\sqrt{b^2-d^2}} \leq 0$ (left, $a = 0.1$, $b = 0.3$, $c = 0.3$, $d = 0.1$) and $d - b + \frac{d(c-a)}{\sqrt{b^2-d^2}} > 0$ (right, $a = 0.1$, $b = 0.3$, $c = 0.3$, $d = 0.2$). Black point $(a, b, \lim_{y \rightarrow b^+} f(a, y))$.

Properties of $\triangle CDE$



Properties of $\triangle CDE$

- ▶ $|PE| = |QD| = \frac{|DE|^2}{|B_0C_0| - |A_0C_0|}$. In particular, this implies that if points D , E , and C_0 are fixed, and point C moves vertically along CC_0 , then lines AB and LM pass through fixed points P and Q , respectively.
- ▶ $|DE| = |A_0B_0| + \frac{[CDE]}{R}$, where $[CDE]$ and R are the area and the circumradius, respectively, of $\triangle CDE$.
- ▶ $\frac{|AA_0|}{|BB_0|} = \frac{|CE|}{|CD|}$.
- ▶ $|CA| + |CB| = \frac{|DE|^2}{|CE| + |CD|}$.
- ▶ $|PQ| = \frac{|DE|^3}{|CE|^2 - |CD|^2}$.
- ▶ $\frac{|CA|}{|CB|} = \frac{|DC_0|}{|EC_0|}$.

Generalization

It is possible to generalize Problem 1 by asking to find the minimum of $(|C_1 D_1|^p + |C_1 E_1|^p)^{\frac{1}{p}}$, which can be interpreted as l_p ($p \geq 1$) norm. We already solved case $p = 1$. We can look at case $p = \infty$. In this case

$$(|C_1 D_1|^p + |C_1 E_1|^p)^{\frac{1}{p}} = \max(|C_1 D_1|, |C_1 E_1|)$$

and the solution is similar.

Problem

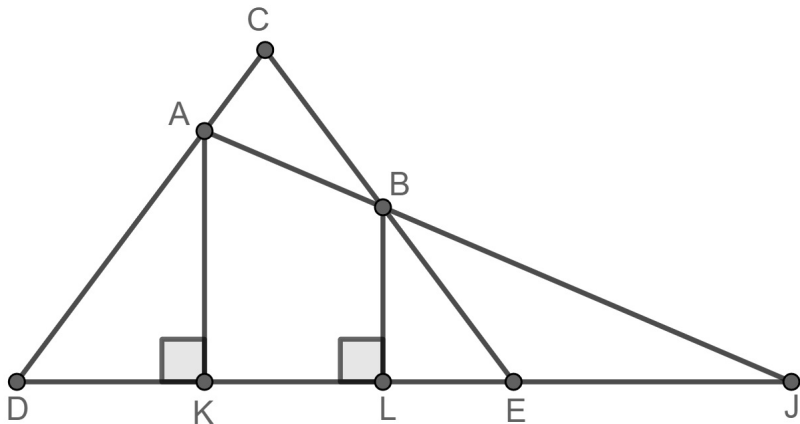
Given $A(a, b)$, $B(c, d)$, with $a < c$ and $b \geq d > 0$ find the minimum of $\max(|C_1 D_1|, |C_1 E_1|)$, where $C_1(x, y)$ is a point such that $y > b$ and rays $C_1 A$ and $C_1 B$ intersect line $y = 0$ at points D_1 and E_1 , respectively.

V. Protasov, V. Tikhomirov, Kvant, 2012, no. 2, 2–11.

Zaslavskii A., Kvant, 2013, no. 5-6, 45–47.

Problema 504, La Gaceta de la RSME, Vol. 27 (2024)

Let A and B be points on sides CD and CE , respectively, of isosceles triangle CDE such that $\tan \angle D = \tan \angle E = \left(\frac{AK+BL}{KL}\right)^{\frac{1}{3}}$, where AK and BL are perpendiculars from points A and B , respectively, to side DE . Let lines AB and DE intersect at point J . Prove that if $AK > BL$, then $AJ \geq CD$



Open problem

I leave as an open question to study the other values of p .
As a problem for further exploration it would also be interesting to study these questions for more than two points instead of just points A and B . One can also replace line DE with a plane and ask the same question for more than two points in space.

Thank you!