Deriving Closed Formulae for the Relative Position of Two Conics

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Motivation

The problem of detecting the collisions or overlap of two moving conics/quadrics is of interest to robotics, CAD/CAM, computational physics, computer animation, computer vision, …, where conics/quadrics are often used for modelling (or enclosing) the shape of the considered objects.

But, we do not want to compute the intersection points since we are sampling many conics/quadrics … .

The problem

There is no real numbers *x* and *y* such that:

$$a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} \leq 0$$
$$b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{10}x + b_{01}y + b_{00} \leq 0$$

G(...,a_{ij},...,b_{ij},...)?

Quantifier Elimination

$$\exists x \in \mathbb{R} \quad x^2 + bx + c = 0$$

$$b^2 - 4c \geq 0$$

There are no real numbers x and y such that:

$$a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} \leq 0$$

$$b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{10}x + b_{01}y + b_{00} \leq 0$$

?

Such a formula exists and there is an algorithm computing it but …

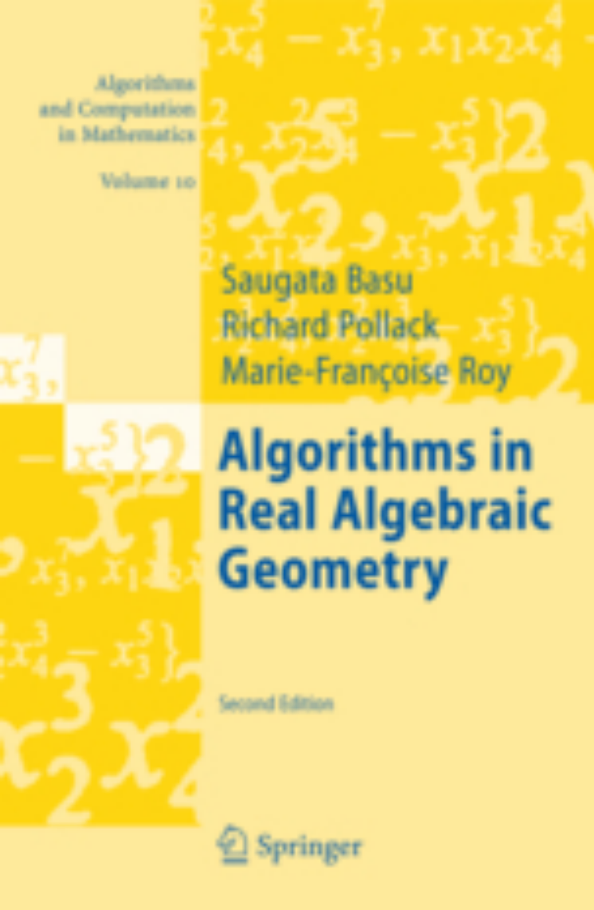
Quantifier Elimination

Quantifier Elimination:

- Formula existence & algorithm.
- Tarski (1951), Seiderberg (1954), Cohen (1969), Kreisel&Krivine (1971), Collins (1976), Hörmander (1983), … .
- General algorithms **very far** from applicable in practice but … .

The underlying theory:

- Semialgebraic sets
- Real Algebraic Geometry



Geometric configurations for

$$a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0$$
$$b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{10}x + b_{01}y + b_{00} = 0$$

G(...,a_{ij},...,b_{ij},...)?

Problem at hand:

- Looking for the ***simplest*** formula in the coefficients of the considered conics certifying each potential geometric configuration.
- To be used when the coefficients depend on parameters.
- ***No computation at all of the intersection points***.

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OUTLINE

- Motivation.
- The case of two coplanar ellipses.
- Interference analysis for general conics.
- Conclusions.

ON THE ELLIPSES INTERFERENCE PROBLEM

The Problem at hand

<u>Goal</u> (first version):
Given two ellipses in 2D, to derive a closed formulae in terms of the coefficients of their defining equations or matrices characterising when they are separated (ie when there exists a line separating them).

Interference analysis for ellipses

The classical theory:

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

$$\begin{pmatrix} x & y & 1 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \qquad \text{with} \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

Given two conics $\mathcal{A} : X^T AX = 0$ and $\mathcal{B} : X^T BX = 0$, their characteristic polynomial is defined by

$$\boxed{f(\lambda) = \det(\lambda A + B) = a\lambda^3 + b\lambda^2 + c\lambda + d}$$

which is a cubic polynomial in λ with real coefficients.

W. Wang, J. Wang, M.-S. Kim: An algebraic condition for the separation of two ellipsoids. Computer Aided Geometric Design 18, 531-539, 2001.

$$f(\lambda) := \det(\lambda A + B)$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two ellipses with characteristic polynomial $f(\lambda)$. Then:

(1) The product of the three roots of the characteristic polynomial is negative.

(2) The characteristic polynomial $f(\lambda)$ has always, at least, one negative real root, and $f(0) \neq 0$.

(3) The characteristic polynomial $f(\lambda)$ has two distinct positive real roots if and only if the two ellipses are separated by a line (case **3**).

(4) The characteristic polynomial $f(\lambda) = 0$ has a positive double root if and only if the two ellipses touch each other externally (case **6**).

(5) $\mathcal{A}$ and $\mathcal{B}$ have a common interior point if and only if all the real roots of the characteristic polynomial $f(\lambda)$ are not positive (cases **1**, **2**, **4**, **5**, **7**, **8**, **9** and **10**).

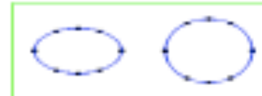
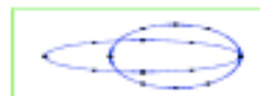
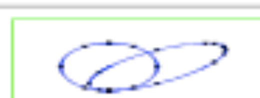
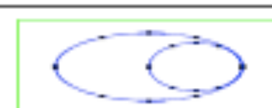
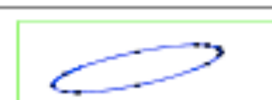
W. Wang, J. Wang, M.-S. Kim: An algebraic condition for the separation of two ellipsoids. Computer Aided Geometric Design 18, 531-539, 2001.

Is the characteristic polynomial enough?

A natural question:

Does the sign of the real roots of the characteristic polynomial characterise (all) the different relative positions of two ellipses?

F. Etayo, L. Gonzalez-Vega, N. Del Rio: A new solution characterising the relative solution of two ellipses. Computer Aided Geometric Design, 23, 324-350, **2006**.

Type of pencils ($\mathbb{P}_2(\mathbb{R})$)	Relative position	Degenerate conics	Roots of $f(\lambda)$
I (no empty conics)	**1** 	3 pairs of real lines	$\alpha < \beta < \gamma < 0$
Ia (with empty conics)	**3** 	1 pair of real lines and 2 pairs of imaginary lines	$\alpha < 0 < \beta < \gamma$
	4 	1 pair of real lines and 2 pairs of imaginary lines	$\alpha < \beta < \gamma < 0$
Ib	**2** 	1 pair of real lines	$\alpha < 0;\ \beta, \overline{\beta} \in \mathbb{C}$
II	**5** 	2 pairs of real lines	$\alpha < 0; \beta, \beta < 0$
IIa	**7** 	1 pair of real lines and 1 pair of imaginary lines	$\alpha < 0; \beta, \beta < 0$
	6 	1 pair of real lines and 1 pair of imaginary lines	$\alpha < 0 < \beta, \beta$
III	**8** 	1 pair of real parallel lines and 1 double real line	$\alpha < 0; \beta, \beta < 0$
IIIa	**4** 	1 pair of imaginary lines and 1 double real line	$\alpha < 0; \beta, \beta < 0$
IV	**9** 	1 pair of real lines	$\alpha, \alpha, \alpha < 0$
V	**7** 	1 double real line	$\alpha, \alpha, \alpha < 0$
VI	**10** 	no degenerate conics	$\alpha, \alpha, \alpha < 0$

The first solution

Let $\mathcal{A} : X^T AX = 0$ and $\mathcal{B} : X^T BX = 0$ be two ellipses and

$$f(\lambda) = \det(\lambda A + B) = \lambda^3 + a\lambda^2 + b\lambda + c$$

their characteristic polynomial (once turned monic).

$\mathcal{A}$ and $\mathcal{B}$ are separated if and only if

$$\begin{array}{llll} a \geq 0, & -3b + a^2 > 0, & 3ac + ba^2 - 4b^2 < 0, & -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0 \quad \text{or} \\ a < 0, & -3b + a^2 > 0, & & -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0. \end{array}$$

F. Etayo, L. Gonzalez-Vega, N. Del Rio: A new solution characterising the relative solution of two ellipses. Computer Aided Geometric Design, 23, 324-350, **2006**.

Separated if and only if

$$18abc + a^2b^2 - 27c^2 - 4a^3c - 4b^3 > 0$$

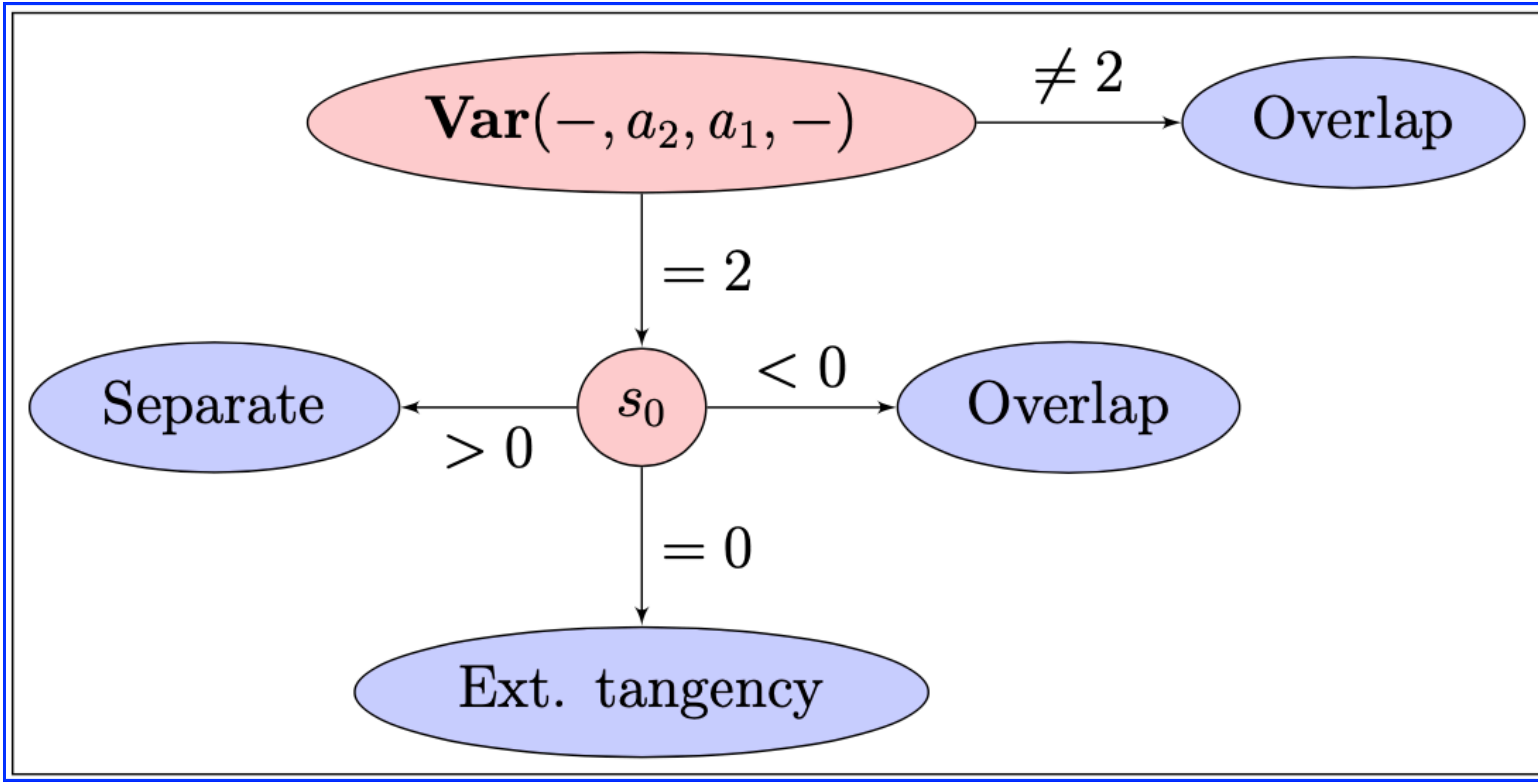
$$a < 0 \quad \text{or} \quad b < 0$$

M. Alberich-Carramiñana, B. Elizalde, F. Thomas: New algebraic conditions for the identification of the relative position of two coplanar ellipses. Computer Aided Geometric Design 54, 35-48, **2017**.

Interference analysis for ellipses

$$P(T) = a_3T^3 + a_2T^2 + a_1T + a_0$$

$$s_0 = a_3\left(27a_0^2a_3^2 - 18a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - a_1^2a_2^2\right)$$



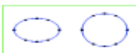
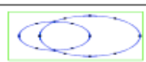
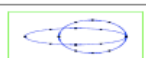
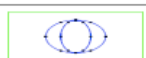
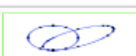
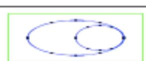
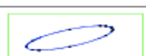
J. Caravantes, G.M. Diaz-Toca, M. Fioravanti, L. Gonzalez-Vega: *Solving the interference problem for ellipses and ellipsoids: new formulae*. Journal of Computational and Applied Mathematics 407, 114072, **2022**.

Interference analysis for ellipses

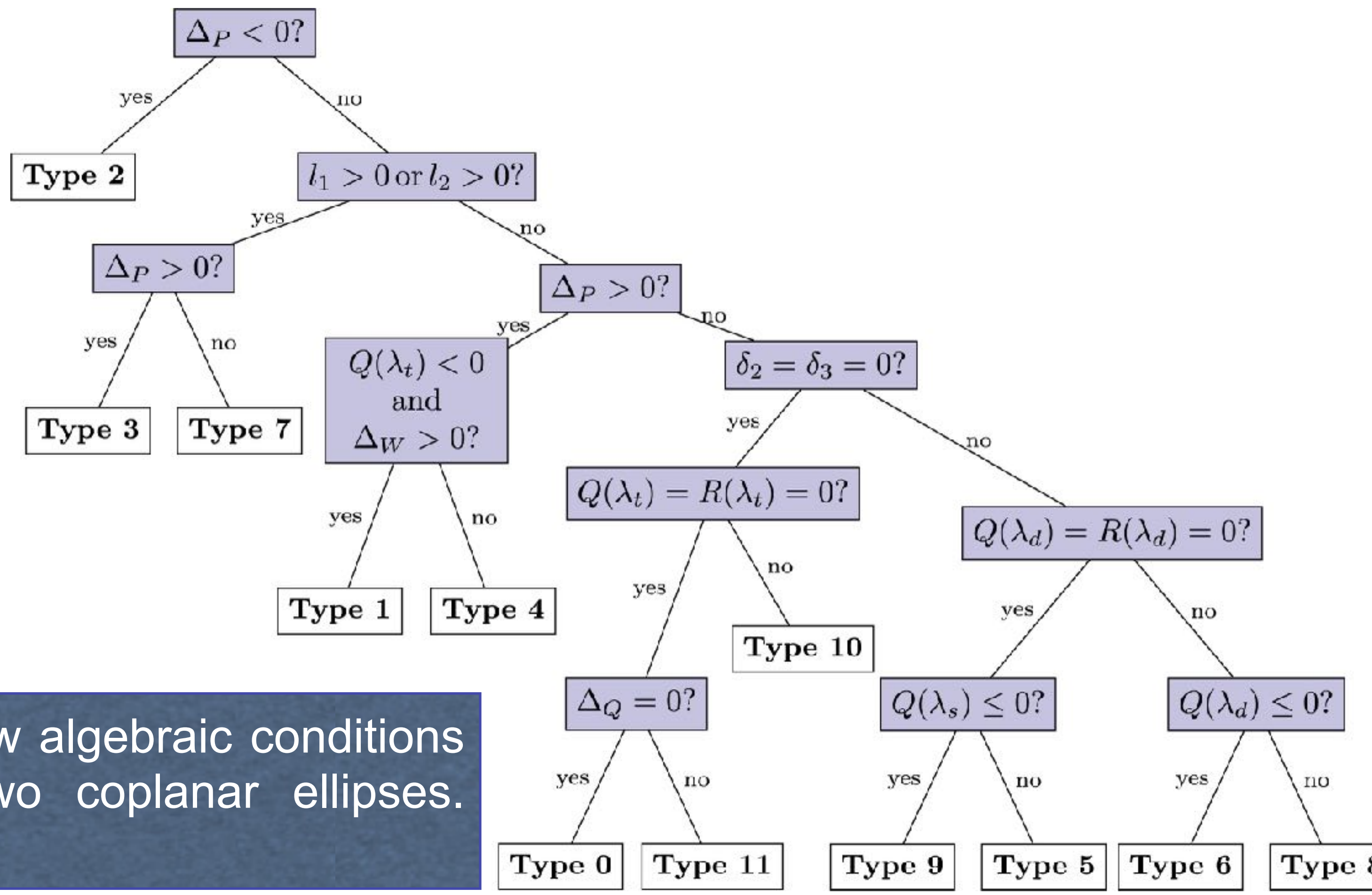
Plane-extraction from depth-data using a Gaussian mixture regression model	Marriott, R.T., Pashevich, A., Horaud, R.	2018	Pattern Recognition Letters 110, pp. 44-50
Shapes within shapes: How particles arrange inside a cavity	Wan, D., Glotzer, S.C.	2018	Soft Matter 14(16), pp. 3012-3017
Astigmatic multipass cell with cylindrical lens	Gupta, A., Udupa, D.V., Topkar, A., Sahoo, N.K.	2017	Journal of Optics (India) 46(3), pp. 324-330
Velocity obstacle based local collision avoidance for a holonomic elliptic robot	Lee, B.H., Jeon, J.D., Oh, J.H.	2017	Autonomous Robots 41(6), pp. 1347-1363
Constraining new resonant physics with top spin polarisation information	Englert, C., Ferrando, J., Nordström, K.	2017	European Physical Journal C 77(6),407
Bisection-based asymmetric M_{T2} computation: a higher precision calculator than existing symmetric methods	Lester, C.G., Nachman, B.	2015	Journal of High Energy Physics 2015(3),100

Interference analysis for conics

Interference analysis for conics: two ellipses

Type of pencils ($\mathbb{P}_2(\mathbb{R})$)	Relative position		Degenerate conics	Roots of $f(\lambda)$
I (no empty conics)	**1**		3 pairs of real lines	$\alpha < \beta < \gamma < 0$
Ia (with empty conics)	**3**		1 pair of real lines and 2 pairs of imaginary lines	$\alpha < 0 < \beta < \gamma$
	4		1 pair of real lines and 2 pairs of imaginary lines	$\alpha < \beta < \gamma < 0$
Ib	**2**		1 pair of real lines	$\alpha < 0;\ \beta, \bar{\beta} \in \mathbb{C}$
II	**5**		2 pairs of real lines	$\alpha < 0; \beta, \beta < 0$
IIa	**7**		1 pair of real lines and 1 pair of imaginary lines	$\alpha < 0; \beta, \beta < 0$
	6		1 pair of real lines and 1 pair of imaginary lines	$\alpha < 0 < \beta, \beta$
III	**8**		1 pair of real parallel lines and 1 double real line	$\alpha < 0; \beta, \beta < 0$
IIIa	**4**		1 pair of imaginary lines and 1 double real line	$\alpha < 0; \beta, \beta < 0$
IV	**9**		1 pair of real lines	$\alpha, \alpha, \alpha < 0$
V	**7**		1 double real line	$\alpha, \alpha, \alpha < 0$
VI	**10**		no degenerate conics	$\alpha, \alpha, \alpha < 0$

F. Etayo, L. Gonzalez-Vega, N. Del Rio: A new solution characterising the relative solution of two ellipses. Computer Aided Geometric Design, 23, 324-350, **2006**.



M. Alberich-Carramiñana, B. Elizalde, F. Thomas: New algebraic conditions for the identification of the relative position of two coplanar ellipses. Computer Aided Geometric Design 54, 35-48, **2017**.

Interference analysis for conics (ellipse and other conic)

Step I.
Circle and Parabola/Hyperbola (in canonical form) configurations are described in terms of the roots of the characteristic equation and other parameters.

Step II. (QE)
Circle and Parabola/Hyperbola configurations are described in terms of the coefficients of the characteristic equation and other parameters.

Step III. (QE & inverting the affine transformation)
Ellipse and Parabola/Hyperbola configurations are described in terms of the coefficients of the characteristic equation of the Ellipse and the Parabola/Hyperbola.

Interference analysis for conics: parabola & ellipse

Parabola $\mathcal{N}$, defined by $\mathcal{X} N \mathcal{X}^t = 0$.
Circle $\mathcal{M}$ of radius δ, defined by $\mathcal{X} M \mathcal{X}^t = 0$.

$$N = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ -x_c & -y_c & -\delta^2 + {x_c}^2 + {y_c}^2 \end{bmatrix}$$

$$\mathcal{X} = [x\ y\ 1]$$

The characteristic equation of N and M is

$$f(\lambda) = \det(\lambda N + M) = c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3.$$

Let Δ be the discriminant of $f(\lambda)$:

$$-27{c_0}^2{c_3}^2 + 18c_0c_1c_2c_3 - 4c_0{c_2}^3 - 4{c_1}^3c_3 + {c_1}^2{c_2}^2$$

Interference analysis for conics: parabola & ellipse

$$\mathscr{M} : \delta, (x_c, y_c)$$

$$\mathscr{N} : \frac{x^2}{a^2} - 2y = 0$$

1. *$\mathcal{M}$ and $\mathcal{N}$ are separated iff $f(\lambda) = 0$ has two distinct positive roots.*

2. *$\mathcal{M}$ and $\mathcal{N}$ are externally tangent iff $f(\lambda) = 0$ has a positive double root.*

3. *$\mathcal{M}$ is inside $\mathcal{N}$ iff $f(\lambda) = 0$ has three distinct negative roots, two of which are not less than $-a^2$ and one root belongs to $(-\infty, -a^2)$, or three roots are $-a^2, -a^2, -\delta^2/a^2$ when $a^2 > \delta$.*

4. *$\mathcal{M}$ and $\mathcal{N}$ have only two intersection points iff $f(\lambda) = 0$ has two imaginary roots.*

5. *$\mathcal{M}$ and $\mathcal{N}$ have four points of intersection iff $f(\lambda) = 0$ has three distinct negative roots which are not greater than $-a^2$.*

Improved

6. *$\mathcal{M}$ and $\mathcal{N}$ have two points of intersection and an inner tangent point iff $f(\lambda) = 0$ has a negative double root different from $-a^2$ and the three roots are not greater than $-a^2$, where $a^2 \leq \delta$.*

7. *$\mathcal{M}$ and $\mathcal{N}$ have only an inner tangent point iff*
   - *$f(\lambda) = 0$ has a negative double root which is greater than $-a^2$, or,*
   - *$-a^2$ is a triple root of $f(\lambda) = 0$.*

8. *$\mathcal{M}$ and $\mathcal{N}$ have two inner tangent points iff the roots of $f(\lambda) = 0$ are $-a^2$,$-a^2$,$-\delta^2/a^2$ where $a^2 < \delta$.*

9. *$\mathcal{M}$ and $\mathcal{N}$ have only an intersection point and an inner tangent point iff $f(\lambda) = 0$ has a triple root, less than $-a^2$.*

Y. Liu, F. Chen: Algebraic Conditions for Classifying the Positional Relationships Between Two Conics and Their Applications. J. Comput. Sci. Technol., 19, 665-673, **2004**.

Interference analysis for conics: parabola & ellipse

$$g(\lambda) := f(\lambda - a^2) = c_0'\lambda^3 + c_1'\lambda^2 + c_2'\lambda + c_3'$$

$$f(\lambda) = \det(\lambda N + M) = c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$$

Consider the parabola $\mathcal{N}$ and the circle $\mathcal{M}$, as above.

1. *$\mathcal{M}$ and $\mathcal{N}$ are separated iff $\Delta > 0$, and $c_1 > 0$ or $c_2 > 0$.*
2. *$\mathcal{M}$ and $\mathcal{N}$ are externally tangent iff $\Delta = 0$, and $c_1 > 0$ or $c_2 > 0$.*
3. *$\mathcal{M}$ is inside $\mathcal{N}$ iff either $\Delta > 0$, $c_1 < 0$, $c_2 < 0$, and the number of sign changes in $-c_0$, c_1', $-c_2'$, c_3' is 1; or $c_3' = 0$, $c_2' = 0$, $a^2 > \delta$.*
4. *$\mathcal{M}$ and $\mathcal{N}$ have only two intersection points iff $\Delta < 0$.*
5. *$\mathcal{M}$ and $\mathcal{N}$ have four points of intersection iff $\Delta > 0, c_2' < 0, c_1' < 0$.*
6. *$\mathcal{M}$ and $\mathcal{N}$ have two points of intersection and an inner tangent point iff $a^2 \le \delta$, $\Delta = 0$, $\Delta' \neq 0$, $c_2' < 0, c_1' < 0$.*
7. *$\mathcal{M}$ and $\mathcal{N}$ have only an inner tangent point iff*
   - *either $\Delta = 0$, $c_1 < 0, c_2 < 0$, $c_3' < 0$, and $c_2' > 0$ or $c_1' > 0$;*
   - *or $\Delta = 0$, $c_1 < 0, c_2 < 0$, $c_3' = 0$, $c_1' > 0$, $c_2' < 0$.*
   - *or $c_1' = c_2' = c_3' = 0$.*
8. *$\mathcal{M}$ and $\mathcal{N}$ have two inner tangent points iff $a^2 < \delta$, $c_3' = 0$, and $c_2' = 0$.*
9. *$\mathcal{M}$ and $\mathcal{N}$ have only an intersection point and an inner tangent point iff $\Delta = 0$, $\Delta' = 0$, $c_1' < 0$ and $c_3' < 0$.*

M. Chen, X. Hou, X. Qiu: An explicit criterion for the positional relationship of an ellipse and a parabola. IEEE International Conference on Systems, Man and Cybernetics (SMC 2008), 825-829, 2008.

Interference analysis for conics: parabola & ellipse

$g(\lambda) := f(\lambda - a^2) = \ldots \lambda^2 + c_2'\lambda + c_3'$

$f(\lambda) = \det(\lambda N + M) \ldots c_1\lambda^2 + c_2\lambda + c_3$

Consider the parabola $\mathcal{N}$ and the circle $\mathcal{M}$, as above.

1. *$\mathcal{M}$ and $\mathcal{N}$ are separated iff $\Delta > 0$, and $c_1 > 0$ or $c_2 > 0$.*
2. *$\mathcal{M}$ and $\mathcal{N}$ are externally tangent iff $\Delta = 0$, and $c_1 > 0$ or $c_2 > 0$.*
3. *$\mathcal{M}$ is inside $\mathcal{N}$ iff either $\Delta > 0$, $c_1 < 0$, $c_2 < 0$, and the number of sign changes in $-c_0$, c_1', $-c_2'$, c_3' is 1; or $c_3' = 0$, $c_2' = 0$, $a^2 > \delta$.*
4. *$\mathcal{M}$ and $\mathcal{N}$ have only two intersection points iff $\Delta < 0$.*
5. *$\mathcal{M}$ and $\mathcal{N}$ have four points of intersection iff $\Delta > 0, c_2' < 0, c_1' < 0$.*

Tools used: Maple, Subresultants and Descartes Law of Signs

- *either $\Delta = 0$, $c_1 < 0, c_2 < 0$, $c_3' < 0$, and $c_2' > 0$ or $c_1' > 0$;*
- *or $\Delta = 0$, $c_1 < 0, c_2 < 0$, $c_3' = 0$, $c_1' > 0$, $c_2' < 0$.*
- *or $c_1' = c_2' = c_3' = 0$.*

8. *$\mathcal{M}$ and $\mathcal{N}$ have two inner tangent points iff $a^2 < \delta$, $c_3' = 0$, and $c_2' = 0$.*
9. *$\mathcal{M}$ and $\mathcal{N}$ have only an intersection point and an inner tangent point iff $\Delta = 0$, $\Delta' = 0$, $c_1' < 0$ and $c_3' < 0$.*

M. Chen, X. Hou, X. Qiu: An explicit criterion for the positional relationship of an ellipse and a parabola. IEEE International Conference on Systems, Man and Cybernetics (SMC 2008), 825-829, 2008.

Interference analysis for conics: parabola & ellipse

$$f(\lambda) = \det(\lambda\mathbf{N} + \mathbf{M}) = L_0\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3 \quad \ldots\ldots, I_4 = 3L_0T_1^2 - 2L_1TT_1 + L_2T^2, \ldots\ldots$$

$$T = T_1\mathrm{Trace}(\mathbf{M_2}^{-1}\mathbf{N_2}), T_1 = \det\mathbf{M_2}, T_2 = \det\mathbf{N_2} \qquad \ldots\ldots, I_5 = L_1T - 3L_0T_1, \ldots\ldots$$

The relationship between the ellipse $\mathcal{M}$ and the parabola $\mathcal{N}$ are as follows:

1. *$\mathcal{M}$ and $\mathcal{N}$ are separated iff $\Delta > 0$ and $(L_1 > 0$ or $L_2 > 0)$.*
2. *$\mathcal{M}$ and $\mathcal{N}$ are externally tangent iff $\Delta = 0$ and $(L_1 > 0$ or $L_2 > 0)$.*
3. *$\mathcal{M}$ is inside $\mathcal{N}$ iff $\{\Delta > 0, L_1 < 0, L_2 < 0, I_4 > 0\}$ or $\{\Delta > 0, L_1 < 0, L_2 < 0, I_5 \geq 0, I_4 \leq 0, I_3 < 0\}$, or $\{I_3 = 0, I_4 = 0, I_5 > 0\}$.*
4. *$\mathcal{M}$ and $\mathcal{N}$ have only two points of intersection iff $\Delta < 0$.*
5. *$\mathcal{M}$ and $\mathcal{N}$ have four points of intersection iff $\Delta > 0$, $I_4 < 0$ and $I_5 < 0$.*
6. *$\mathcal{M}$ and $\mathcal{N}$ have one inner tangent point and two points of intersection points iff $I_2 \leq 0, \Delta = 0, \Delta' \neq 0, I_4 < 0, I_5 < 0$.*
7. *$\mathcal{M}$ and $\mathcal{N}$ have only one inner tangent point iff $\{\Delta = 0,\ L_1 < 0,\ L_2 < 0,\ I_3 < 0,$ $(I_4 > 0$ or $I_5 > 0)\}$, or $\{\Delta = 0,\ L_1 < 0,\ L_2 < 0,\ I_3 = 0,\ I_5 > 0,\ I_4 < 0\}$, or $\{I_3 = 0,$ $I_5 = 0,\ I_4 = 0\}$.*
8. *$\mathcal{M}$ and $\mathcal{N}$ have only two inner points of tangency points iff $I_3 = 0$, $I_4 = 0$ and $I_5 < 0$.*
9. *$\mathcal{M}$ and $\mathcal{N}$ have only an intersection point and an inner tangent point iff $\Delta = 0$, $\Delta' = 0$, $I_3 < 0$ and $I_5 < 0$.*

$\mathscr{M}$: ellipse
$\mathscr{N}$: parabola

M. Chen, X. Hou, X. Qiu: An explicit criterion for the positional relationship of an ellipse and a parabola. IEEE International Conference on Systems, Man and Cybernetics (SMC 2008), 825-829, 2008.

Interference analysis for conics: parabola & ellipse

$$f(\lambda) = \det(\lambda\mathbf{N} + \mathbf{M}) = L_0\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3 \quad \ldots\ldots, I_4 = 3L_0T_1^2 - 2L_1TT_1 + L_2T^2, \ldots\ldots$$

$$T = T_1\mathrm{Trace}(\mathbf{M_2}^{-1}\mathbf{N_2}), T_1 = \det\mathbf{M_2}, T_2 = \det\mathbf{N_2} \qquad \ldots\ldots, I_5 = L_1T - 3L_0T_1, \ldots\ldots$$

The relationship between the ellipse $\mathcal{M}$ and the parabola $\mathcal{N}$ are as follows:

1. *$\mathcal{M}$ and $\mathcal{N}$ are separated iff $\Delta > 0$ and $(L_1 > 0$ or $L_2 > 0)$.*
2. *$\mathcal{M}$ and $\mathcal{N}$ are externally tangent iff $\Delta = 0$ and $(L_1 > 0$ or $L_2 > 0)$.*
3. *$\mathcal{M}$ is inside $\mathcal{N}$ iff $\{\Delta > 0, L_1 < 0, L_2 < 0, I_4 > 0\}$ or $\{\Delta > 0, L_1 < 0, L_2 < 0, I_5 \geq 0, I_4 \leq 0, I_3 < 0\}$, or $\{I_3 = 0, I_4 = 0, I_5 > 0\}$.*
4. *$\mathcal{M}$ and $\mathcal{N}$ have only two points of intersection iff $\Delta < 0$.*

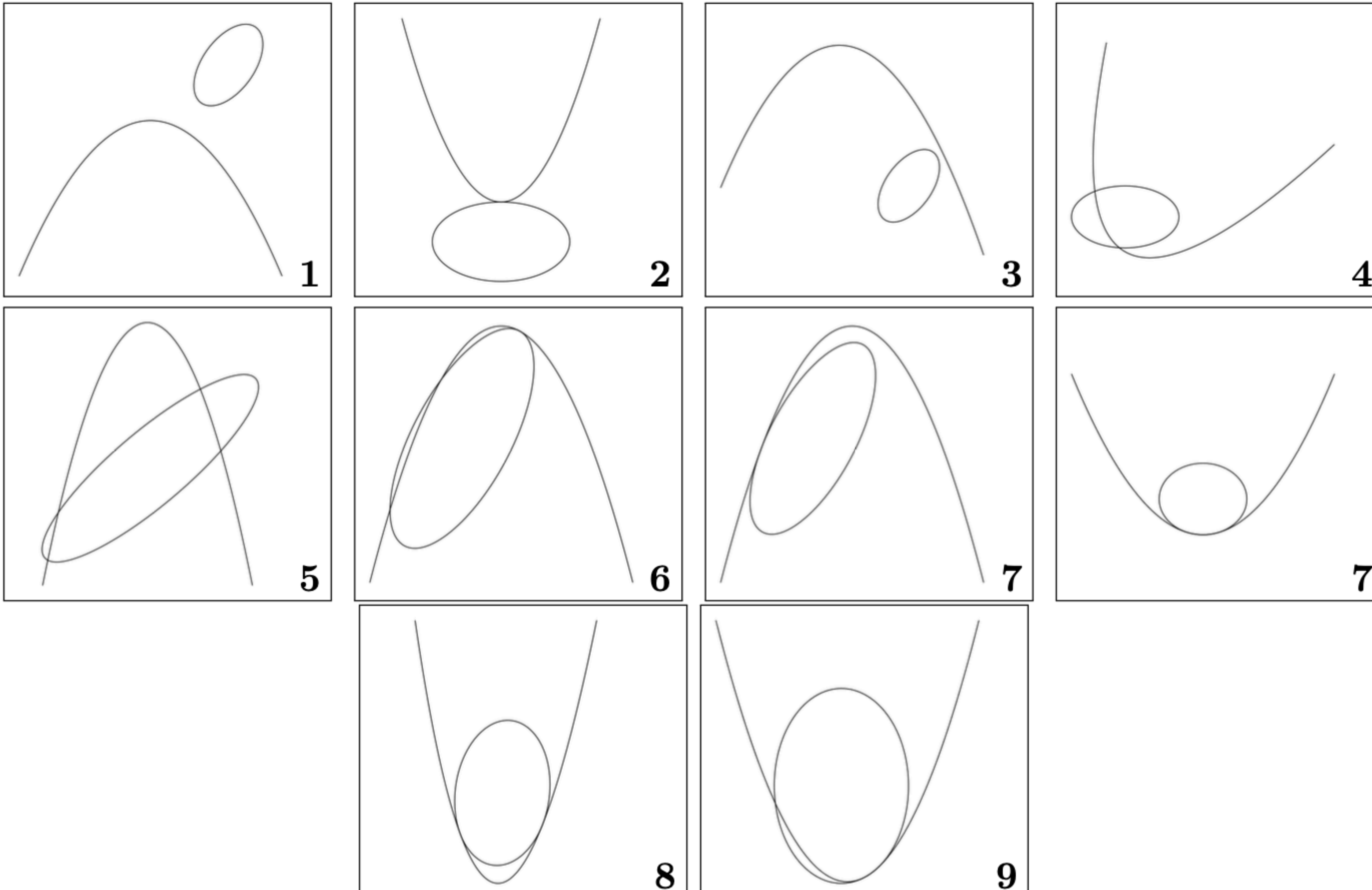
In most cases, we have produced less sign conditions involving a smaller set of polynomials.

$I_5 = 0,\ I_4 = 0\}$.

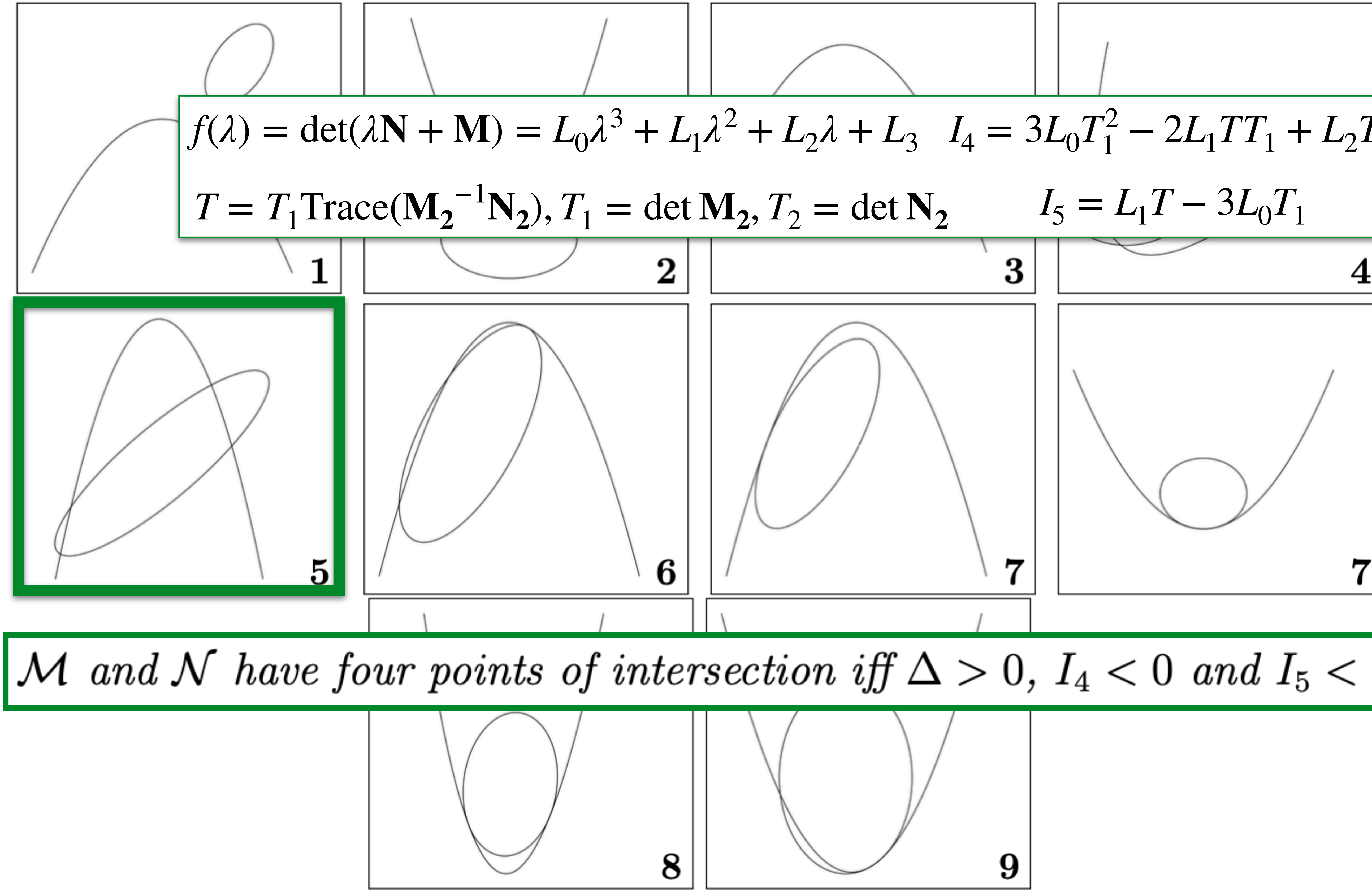
8. *$\mathcal{M}$ and $\mathcal{N}$ have only two inner points of tangency points iff $I_3 = 0$, $I_4 = 0$ and $I_5 < 0$.*
9. *$\mathcal{M}$ and $\mathcal{N}$ have only an intersection point and an inner tangent point iff $\Delta = 0$, $\Delta' = 0$, $I_3 < 0$ and $I_5 < 0$.*

M. Chen, X. Hou, X. Qiu: An explicit criterion for the positional relationship of an ellipse and a parabola. IEEE International Conference on Systems, Man and Cybernetics (SMC 2008), 825-829, 2008.

Interference analysis for conics: parabola & ellipse



Interference analysis for conics: parabola & ellipse



$$f(\lambda) = \det(\lambda \mathbf{N} + \mathbf{M}) = L_0\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3 \qquad I_4 = 3L_0T_1^2 - 2L_1TT_1 + L_2T^2$$

$$T = T_1\,\mathrm{Trace}(\mathbf{M_2}^{-1}\mathbf{N_2}),\ T_1 = \det \mathbf{M_2},\ T_2 = \det \mathbf{N_2} \qquad I_5 = L_1T - 3L_0T_1$$

$\mathcal{M}$ and $\mathcal{N}$ have four points of intersection iff $\Delta > 0$, $I_4 < 0$ *and* $I_5 < 0$

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Interference analysis for conics: hyperbola & ellipse

Hyperbola $\mathcal{H} : \mathcal{X}\mathbf{H}\mathcal{X}^t = \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$ with

$$\mathbf{H} = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & -b^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a, b > 0$$

Circle $\mathcal{M} : \mathcal{X}\mathbf{M}\mathcal{X}^t = 0$ with

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ -x_c & -y_c & -\delta^2 + {x_c}^2 + {y_c}^2 \end{bmatrix}$$

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Interference analysis for conics: hyperbola & ellipse

$$\mathscr{M} : \delta, (x_c, y_c)$$

$$\mathscr{H} : \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

1. *$\mathcal{H}$ and $\mathcal{M}$ are separated iff $f(\lambda) = 0$ has two distinct positive roots equal to or less than b^2; that is, $\lambda_1 < 0 < \lambda_2 < \lambda_3 \leq b^2$.*

2. *$\mathcal{H}$ and $\mathcal{M}$ have only two points of intersection iff $f(\lambda) = 0$ has two imaginary roots.*

Improved

3. *$\mathcal{H}$ and $\mathcal{M}$ have only an inner tangent point iff $f(\lambda) = 0$ has a negative double root greater than $-a^2$ ($\lambda_1 < -a^2 < \lambda_2 = \lambda_3 < 0$), or the three roots are $-a^2$, $-b\delta$, $-b\delta$ with $a^2 \geq b\delta$.*

4. *$\mathcal{H}$ and $\mathcal{M}$ have only two inner tangent points iff $a^2 < b\delta$ and the roots of $f(\lambda)$ are $-a^2$, $-a^2$, $-\frac{b^2\delta^2}{a^2}$.*

5. *$\mathcal{H}$ and $\mathcal{M}$ have two points of intersection and an inner tangent point iff $f(\lambda) = 0$ has a negative double root and all roots are less than or equal to $-a^2$ with $a^2 < b\delta$. If a root is $-a^2$, then the three roots are $-a^2$,$-b\delta$, $-b\delta$.*

6. *$\mathcal{H}$ and $\mathcal{M}$ have only one outer tangent point iff $f(\lambda) = 0$ has a positive double root which is less than b^2; that is $\lambda_1 < 0 < \lambda_2 = \lambda_3 < b^2$.*

7. *$\mathcal{H}$ and $\mathcal{M}$ have two outer tangent points iff the roots of $f(\lambda) = 0$ are $b^2, b^2, -\frac{a^2\delta^2}{b^2}$ with $b \leq \delta$.*

8. *$\mathcal{H}$ and $\mathcal{M}$ have two points of intersection and an outer tangent point iff $f(\lambda)$ has a positive double root greater than b^2, $b^2 < \lambda_1 = \lambda_2$, with $b < \delta$.*

9. *$\mathcal{H}$ and $\mathcal{M}$ have four points of intersection iff either $f(\lambda) = 0$ has three distinct negative roots which are not greater than $-a^2$, $\lambda_1 < \lambda_2 < \lambda_3 \leq -a^2$, with $a^2 < b\delta$ or $f(\lambda)$ has two distinct positive roots not less than b^2, $\lambda_1 < 0 < b^2 \leq \lambda_2 < \lambda_3$ with $b < \delta$.*

10. *$\mathcal{M}$ is inside $\mathcal{H}$ iff $f(\lambda) = 0$ has three distinct negative roots, two of which are not less than $-a^2$,($\lambda_1 < -a^2 \leq \lambda_2 < \lambda_3 < 0$ or $\lambda_1 \leq -a^2 < \lambda_2 < \lambda_3 < 0$), or the three roots are $-a^2$, $-a^2$, $-\frac{b^2\delta^2}{a^2}$ with $a^2 > b\delta$.*

11. *$\mathcal{M}$ and $\mathcal{H}$ have only an intersection point and an inner tangent point iff $f(\lambda) = 0$ has a triple negative root less than $-a^2$.*

Y. Liu, F. Chen: Algebraic Conditions for Classifying the Positional Relationships Between Two Conics and Their Applications. J. Comput. Sci. Technol., 19, 665-673, 2004.

Interference analysis for conics: hyperbola & ellipse

$$q(\lambda) = f(\lambda + b^2) = a_3''\lambda^3 + a_2''\lambda^2 + a_1''\lambda + a_0''$$

$$g(\lambda) = f(\lambda - a^2) = a_3'\lambda^3 + a_2'\lambda^2 + a_1'\lambda + a_0'$$

$$f(\lambda) = \det(\lambda N + M) = c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$$

1. $\mathcal{H}$ and $\mathcal{M}$ are separated iff $\Delta > 0, \mathbf{Var}(f) = 2, \mathbf{Var}(q) = 0$.
2. $\mathcal{H}$ and $\mathcal{M}$ have only two points of intersection iff $\Delta < 0$.
3. $\mathcal{H}$ and $\mathcal{M}$ have only an inner tangent point iff either $\Delta = 0$, $\mathbf{Var}(f) = 0$, $\mathbf{Var}(g) = 2$ and $a_0' \neq 0$, or $a_0' = 0$, $a_2' \geq 0$ and $|y_c| = b + \delta$.
4. $\mathcal{H}$ and $\mathcal{M}$ have only two inner tangent points iff $a_0' = 0$, $a_1' = 0$ and $a_2' < 0$.
5. $\mathcal{H}$ and $\mathcal{M}$ have two points of intersection and an inner tangent point iff
   - either $\Delta = 0$, $\Delta' \neq 0$, $\mathbf{Var}(g) = 0$ and $a_0' \neq 0$,
   - or $a_0' = 0$, $|y_c| = b + \delta$ and $a_2' < 0$.
6. $\mathcal{H}$ and $\mathcal{M}$ have only one outer tangent point iff $\Delta = 0$, $\mathbf{Var}(f) = 2$, $\mathbf{Var}(q) = 0$, $y_c \neq 0$.
7. $\mathcal{H}$ and $\mathcal{M}$ have two outer tangent points iff $a_0'' = 0, a_1'' = 0$.
8. $\mathcal{H}$ and $\mathcal{M}$ have two points of intersection and an outer tangent point if $\Delta = 0$, $\mathbf{Var}(q) = 2, a_0'' \neq 0$.
9. $\mathcal{H}$ and $\mathcal{M}$ have four points of intersection iff $\Delta > 0$ and either $\mathbf{Var}(g) = 0$; or $\mathbf{Var}(q) > 0$.
10. $\mathcal{M}$ is inside $\mathcal{H}$ iff $\mathbf{Var}(f) = 0$ and, either $\Delta > 0$, $a_0' \neq 0$, $\mathbf{Var}(g) = 2$; or $\Delta > 0$, $a_0' = 0$, $a_1' \neq 0$ and $\mathbf{Var}(g) > 0$; or $a_0' = 0$, $a_1' = 0$ and $a_2' > 0$.
11. $\mathcal{M}$ and $\mathcal{H}$ have only an intersection point and an inner tangent point iff $\Delta = 0$, $\Delta' = 0$, $a_0' < 0$ and $a_2' < 0$.

Interference analysis for conics: hyperbola & ellipse

$q(\lambda) = f(\lambda + b^2) =$ … $a_2''\lambda^2 + a_1''\lambda + a_0''$

$g(\lambda) = f(\lambda - a^2) =$ … $a_2'\lambda^2 + a_1'\lambda + a_0'$

$f(\lambda) = \det(\lambda N + M)$ … $c_1\lambda^2 + c_2\lambda + c_3$

1. $\mathcal{H}$ and $\mathcal{M}$ are separated iff $\Delta > 0$, $\mathbf{Var}(f) = 2$, $\mathbf{Var}(q) = 0$.

2. $\mathcal{H}$ and $\mathcal{M}$ have only two points of intersection iff $\Delta < 0$.

3. $\mathcal{H}$ and $\mathcal{M}$ have only an inner tangent point iff either $\Delta = 0$, $\mathbf{Var}(f) = 0$, $\mathbf{Var}(g) = 2$ and $a_0' \neq 0$, or $a_0' = 0$, $a_2' \geq 0$ and $|y_c| = b + \delta$.

4. $\mathcal{H}$ and $\mathcal{M}$ have only two inner tangent points iff $a_0' = 0$, $a_1' = 0$ and $a_2' < 0$.

5. $\mathcal{H}$ and $\mathcal{M}$ have two points of intersection and an inner tangent point iff
   - either $\Delta = 0$, $\Delta' \neq 0$, $\mathbf{Var}(g) = 0$ and $a_0' \neq 0$,

Tools used: Maple, Subresultants and Descartes Law of Signs

8. $\mathcal{H}$ and $\mathcal{M}$ have two points of intersection and an outer tangent point iff $\Delta = 0$, $\mathbf{Var}(q) = 2, a_0'' \neq 0$.

9. $\mathcal{H}$ and $\mathcal{M}$ have four points of intersection iff $\Delta > 0$ and either $\mathbf{Var}(g) = 0$; or $\mathbf{Var}(q) > 0$.

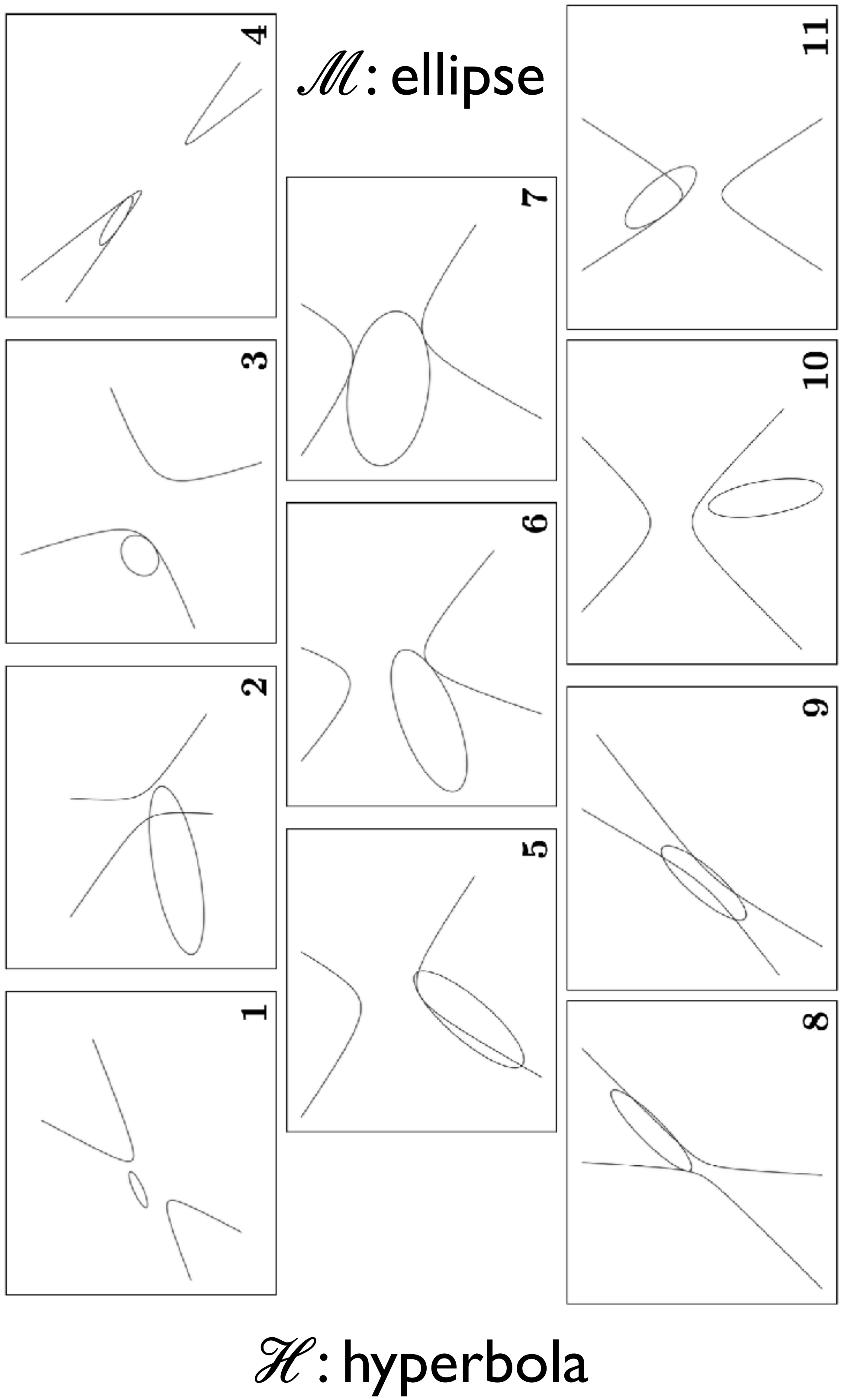
10. $\mathcal{M}$ is inside $\mathcal{H}$ iff $\mathbf{Var}(f) = 0$ and, either $\Delta > 0$, $a_0' \neq 0$, $\mathbf{Var}(g) = 2$; or $\Delta > 0$, $a_0' = 0$, $a_1' \neq 0$ and $\mathbf{Var}(g) > 0$; or $a_0' = 0$, $a_1' = 0$ and $a_2' > 0$.

11. $\mathcal{M}$ and $\mathcal{H}$ have only an intersection point and an inner tangent point iff $\Delta = 0$, $\Delta' = 0$, $a_0' < 0$ and $a_2' < 0$.

Interference analysis for conics: hyperbola & ellipse

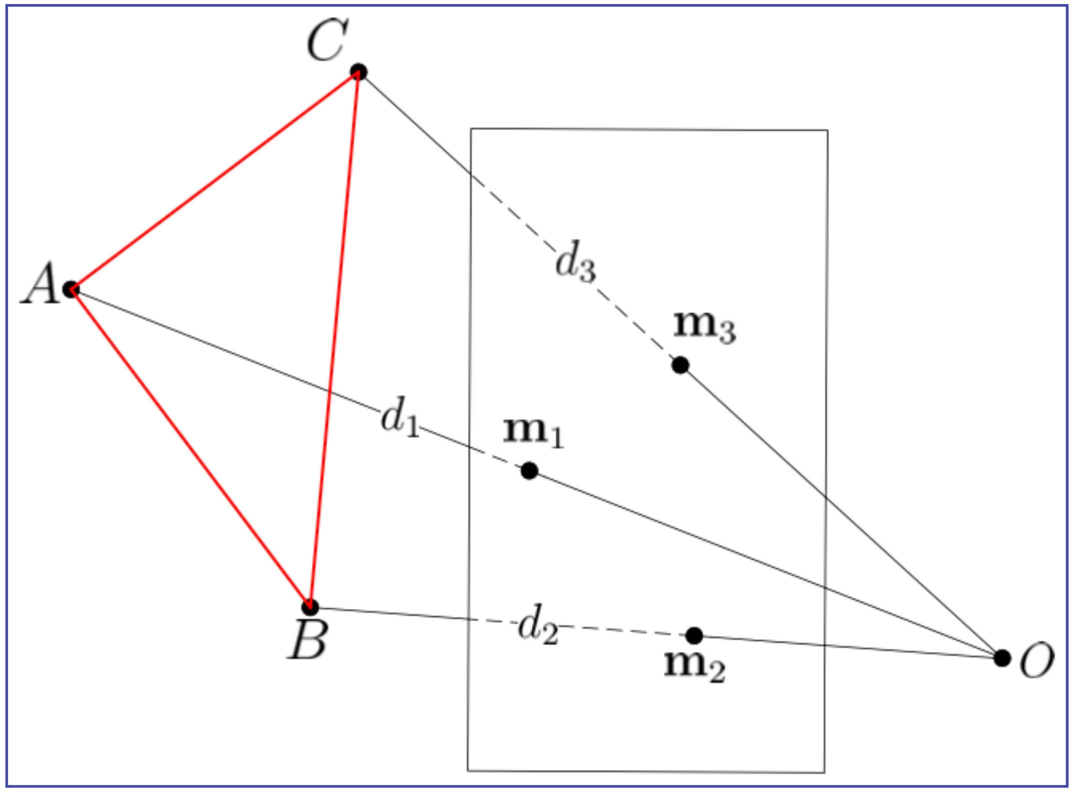
$$F(\lambda) = \det(\lambda \mathbf{N} + \mathbf{M}) = L_0\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3$$

1. *$\mathcal{H}$ and $\mathcal{M}$ are separated iff* $\Delta > 0, \mathbf{Var}(F) = 2, \mathbf{Var}(L_0, K_5, K_4, K_3) = 0$.
2. *$\mathcal{H}$ and $\mathcal{M}$ have only two points of intersection iff* $\Delta < 0$.
3. *$\mathcal{H}$ and $\mathcal{M}$ have only an inner tangent point iff either* $\Delta = 0$, $\mathbf{Var}(F) = 0$, $J_3 \neq 0$, $\mathbf{Var}(L_0, J_5, J_4, J_3) = 2$, *or* $J_3 = 0$, $J_4 = 0$, $J_5 = 0$, *or* $J_1 = 0$, $J_3 = 0$, $J_4 \neq 0$ *and* $J_5 > 0$.
4. *$\mathcal{H}$ and $\mathcal{M}$ have only two inner tangent points iff* $J_3 = 0$, $J_4 = 0$ *and* $J_5 < 0$.
5. *$\mathcal{H}$ and $\mathcal{M}$ have two points of intersection and an inner tangent point iff either* $\Delta = 0$, $\Delta' \neq 0$, $\mathbf{Var}(L_0, J_5, J_4, J_3) = 0, J_3 \neq 0$; *or* $\Delta = 0$, $\mathbf{Var}(L_0, J_5, J_4) = 0, J_3 = 0$, $J_5 < 0$ *and* $J_1 = 0$.
6. *$\mathcal{H}$ and $\mathcal{M}$ have only one outer tangent point iff* $\Delta = 0$, $\mathbf{Var}(F) = 2$, $\mathbf{Var}(L_0, K_5, K_4, K_3) = 0$ *and* $K_3 \neq 0$.
7. *$\mathcal{H}$ and $\mathcal{M}$ have two outer tangent points iff* $K_3 = 0, K_4 = 0$.
8. *$\mathcal{H}$ and $\mathcal{M}$ have two points of intersection and an outer tangent point if* $\Delta = 0$, $\mathbf{Var}(L_0, K_5, K_4, K_3) = 2, K_3 \neq 0$.
9. *$\mathcal{H}$ and $\mathcal{M}$ have four points of intersection iff either* $\Delta > 0$ *and* $\mathbf{Var}(L_0, J_5, J_4, J_3) = 0$ *or* $\Delta > 0$ *and* $\mathbf{Var}(L_0, K_5, K_4, K_3) > 0$.
10. *$\mathcal{M}$ is inside $\mathcal{H}$ iff either* $\Delta > 0$, $\mathbf{Var}(F) = 0$, $J_3 \neq 0$, $\mathbf{Var}(L_0, J_5, J_4, J_3) = 2$ *or* $\Delta > 0$, $\mathbf{Var}(F) = 0$, $J_3 = 0$, $J_4 \neq 0$, $\mathbf{Var}(L_0, J_5, J_4) > 0$; *or* $J_3 = 0$, $J_4 = 0$, $J_5 > 0$.
11. *$\mathcal{M}$ and $\mathcal{H}$ have only an intersection point and an inner tangent point iff* $\Delta = 0$, $\Delta' = 0$, $J_3 < 0$ *and* $J_5 < 0$.



Interference analysis for conics: an application

The perspective-three-point problem



$$x^2 + (1-a)y^2 - 2m_{12}xy + 2am_{23}y - a = 0,$$
$$x^2 - by^2 - 2m_{13}x + 2bm_{23}y + 1 - b = 0,$$

where

$$a = |AB|^2/|BC|^2,\ b = |AC|^2/|BC|^2,$$
$$m_{12} = \mathbf{m}_1^\top \mathbf{m}_2,\ m_{13} = \mathbf{m}_1^\top \mathbf{m}_3,\ m_{23} = \mathbf{m}_2^\top \mathbf{m}_3.$$

While the current state-of-the-art solvers are both extremely fast and stable, there still exist configurations where they break down.

The introduced formulae allow to determine in advance the conics configuration to deal with in terms of the parameters a, b and $m_{i,j}$.

Y. Ding, J. Yang, V. Larsson, C. Olsson, K. Åström: Revisiting the P3P Problem. Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (CVPR), 4872-4880, 2023.

CONCLUSIONS

Conclusions

Closed form solutions try to concentrate at the very end the application of numerical techniques.

Closed form solutions for answering queries about geometric entities are proven to be an approach with many applications in very different contexts.

Quantifier Elimination, with the help of Maple, is a useful tool to derive closed form solutions for quadrics and conics interference problems.

https://arxiv.org/abs/2505.00706

Thanks !

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https://arxiv.org/abs/2505.00706

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