Maple Conference

EXPLORING THE FRONTIERS OF MATHEMATICS TECHNOLOGY

Addressing, with the concourse of Maple, diverse open issues dealing with the manifold and subtle concept of geometric locus

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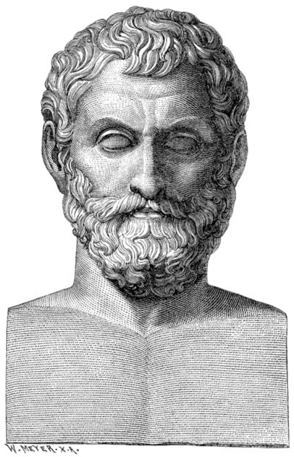
Maple Transactions

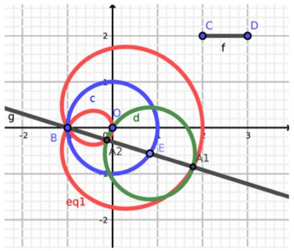


From Thales Theorem to octic curves:

An illustrative journey, with the concourse of Computer Algebra and Dynamic Geometry software, through the uncertain territory of locus computation

Tomas Recio, Thierry Dana-Picard

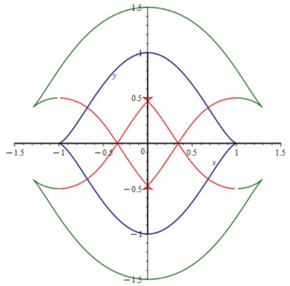
2024-10-31



Dynamic and automated constructions of plane curves

Thierry Dana-Picard, Zoltan Kovacs

2023-11-02



Envelopes of Circles Centered on a Kiss Curve

Thierry Dana-Picard, Daniel Tsirkin

2025-08-01

- Locus computation is a characteristic, defining feature of Dynamic Geometry (DG) software:

> "There is a wide consensus among DG developers to consider locus computation as one of the five basic properties in the DG paradigm (together with dynamic transformation, measurement, free dragging and animation) (X. S. Gao, 1999)"*

- Already present, over sixty years ago, in the precursor of current DG programs, *SketchPad*

(*) Abanades, M.; Botana, F.; Montes, A.; Recio, T. (2014). An algebraic taxonomy for locus computation in dynamic geometry. Computer-Aided Design 56 , 22-33. DOI: 10.1016/j.cad.2014.06.008

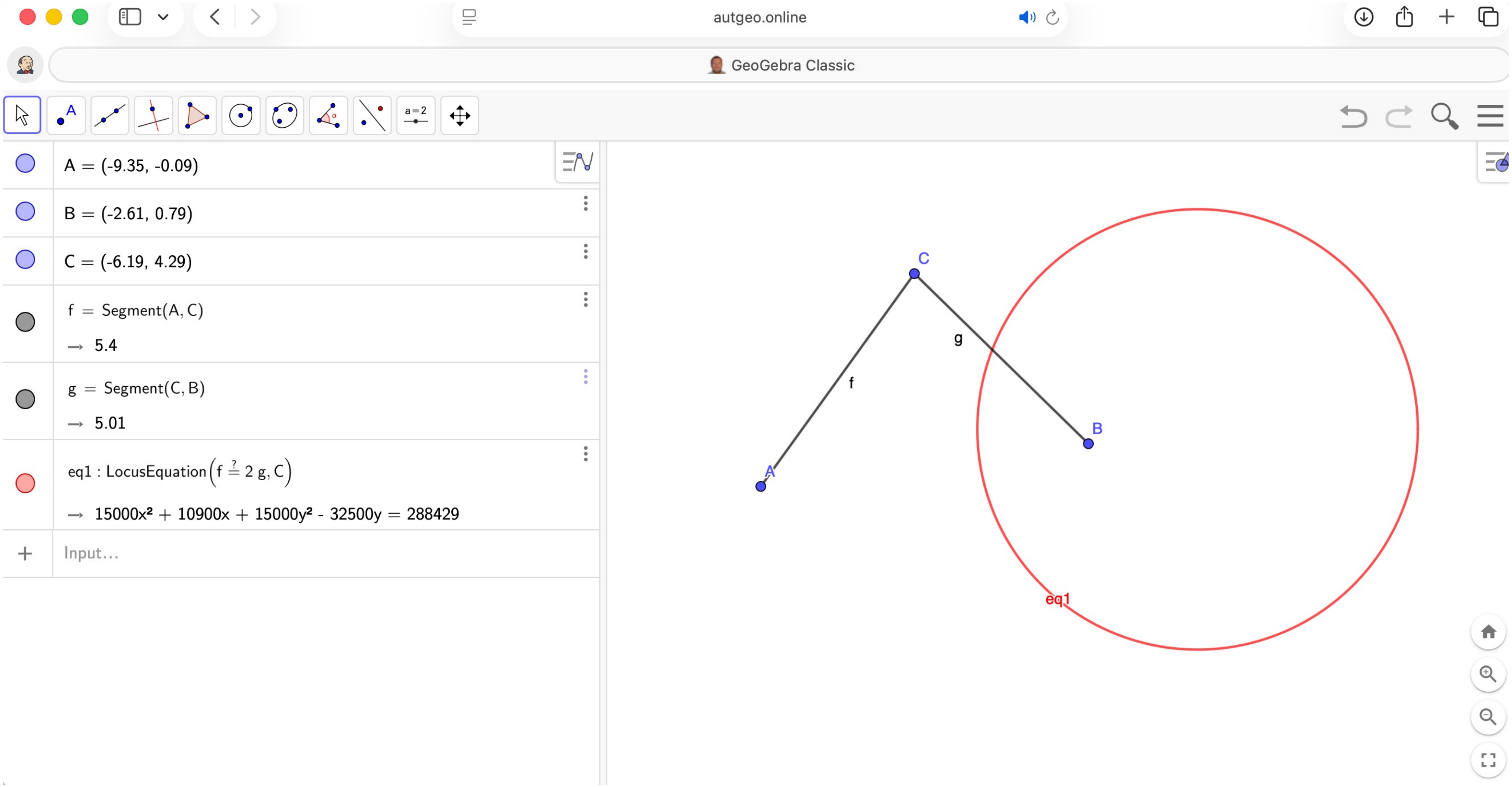
X. S. Gao, Automated geometry diagram construction and engineering geometry, in: X. S. Gao, D. Wang, L. Yang (Eds.), ADG 1998, Vol. 1669700 of Lecture Notes in Artificial Intelligence, Springer, Heidelberg, 1999, pp. 232–257

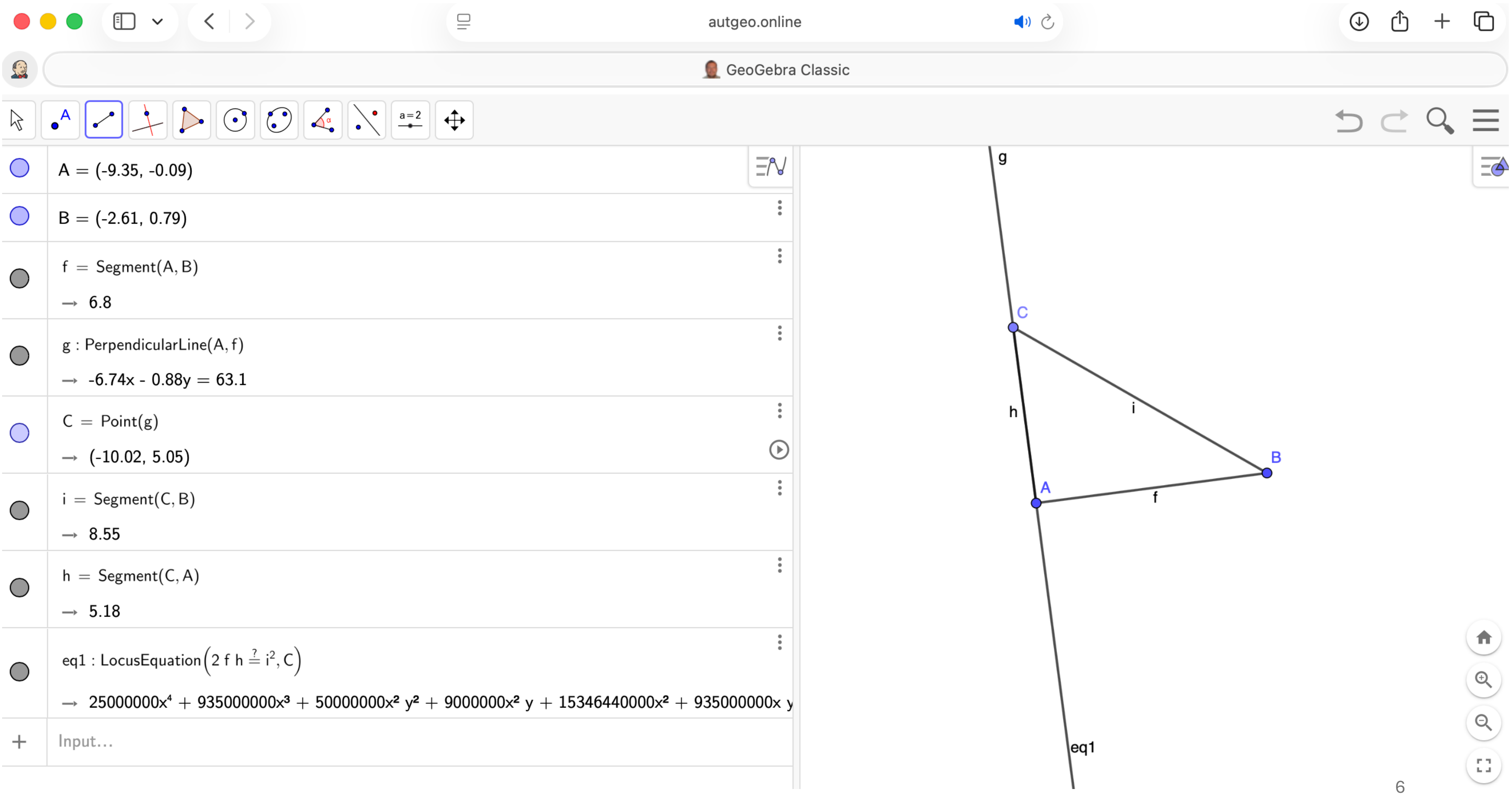
Yet, current locus computation commands in DG programs are either

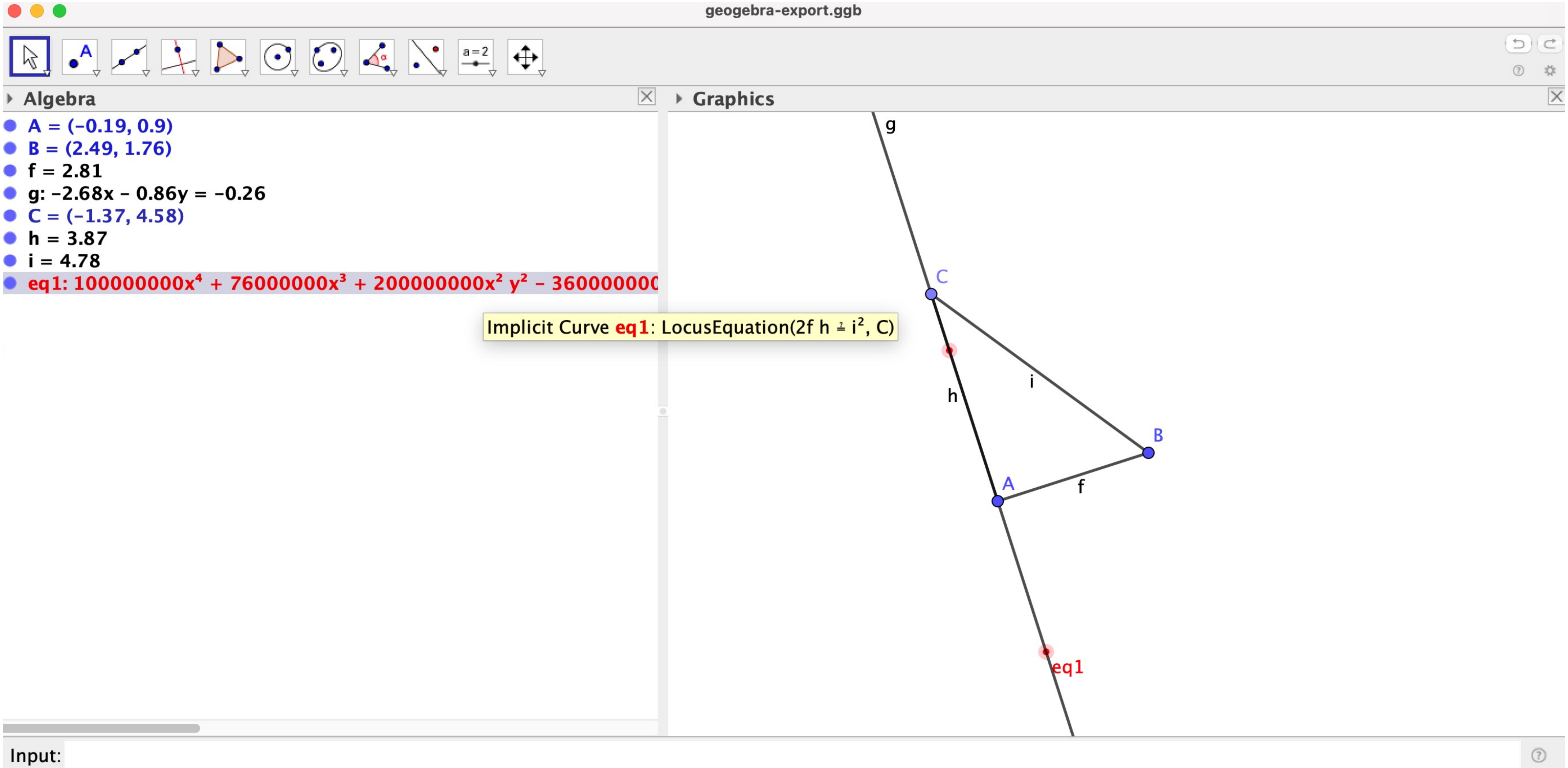
just visual plots (mover---tracer)

just approximate, numerical equations

There is NO symbolic approach to locus computation, even in DG programs that include Computer Algebra features, such as GeoGebra


autgeo.online
GeoGebra Classic
a=2
A = (-9.35, -0.09)
B = (-2.61, 0.79)
C = (-6.19, 4.29)
f = Segment(A, C)
→ 5.4
g = Segment(C, B)
→ 5.01
eq1 : LocusEquation(f ≟ 2 g, C)
→ 15000x² + 10900x + 15000y² - 32500y = 288429
Input...
C
g
f
B
A
eq1


autgeo.online
GeoGebra Classic
a = 2
A = (-9.35, -0.09)
B = (-2.61, 0.79)
f = Segment(A, B)
→ 6.8
g : PerpendicularLine(A, f)
→ -6.74x - 0.88y = 63.1
C = Point(g)
→ (-10.02, 5.05)
i = Segment(C, B)
→ 8.55
h = Segment(C, A)
→ 5.18
eq1 : LocusEquation(2 f h ≟ i², C)
→ 25000000x⁴ + 935000000x³ + 50000000x² y² + 9000000x² y + 15346440000x² + 935000000x y
Input…
g
C
h
i
B
A
f
eq1


geogebra-export.ggb
a=2
Algebra
A = (−0.19, 0.9)
B = (2.49, 1.76)
f = 2.81
g: −2.68x − 0.86y = −0.26
C = (−1.37, 4.58)
h = 3.87
i = 4.78
eq1: 100000000x⁴ + 76000000x³ + 200000000x² y² − 36000000(
Implicit Curve eq1: LocusEquation(2f h ≟ i², C)
Graphics
g
C
i
h
B
A
f
eq1
Input:

FINAL GOAL:

the future consideration and implementation of a symbolic algorithmic approach to locus computation in GeoGebra Discovery that would allow a sound, automated verification of the correction and of the interpretation of its output

OUR ONGOING WORK:

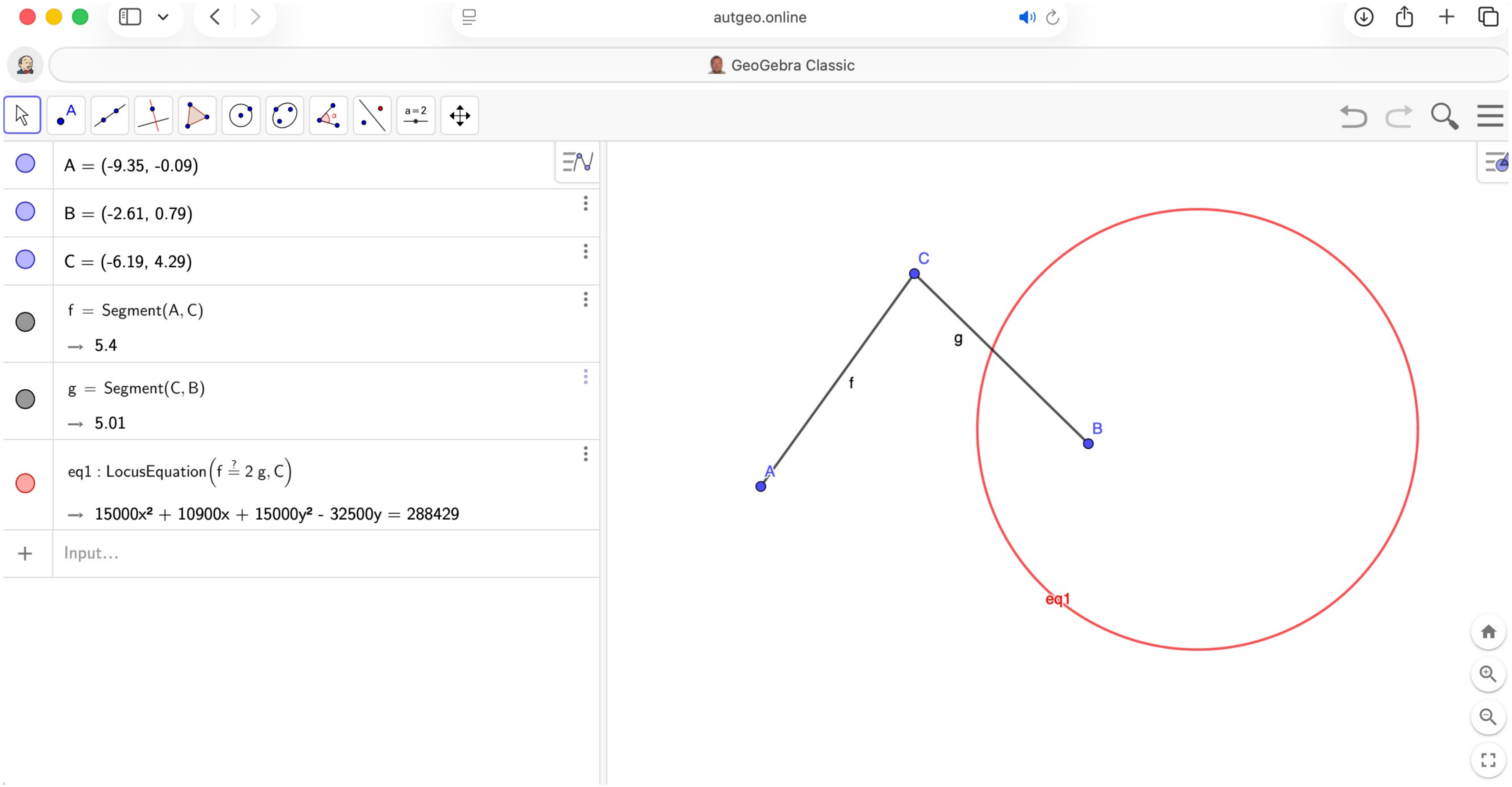
experimenting with Maple the different algorithmic possibilities to implement a symbolic locus equation command

INVOLVED SYMBOLIC COMPUTATION ISSUES:

- How to deal with symbolic locus, i.e. with locus defined in the context of geometric constructions with arbitrary (parametric?) coordinates, and how to deal with the corresponding specialization for numerical values of the coordinates of the basic points of the same construction?
- Should we consider, in the EliminationIdeal protocol that is usually associated to locus computation, the parameters as variables or as elements of the field of coefficients?
- Should we eliminate over the coordinates of the locus point, or over a set of independent variables of the ideal describing the geometric construction where the locus point stands in?
- In what sense will we "discover" geometric statements by looking for the locus of some point in a construction that verifies a certaint constraint?

- **Lemma 2.3.** Given an ideal $\mathfrak{a} \subset K[p,x,y]$, for some sets of variables {p,x,y}, let $\mathfrak{b}$ be the elimination ideal $\mathfrak{a}K[p,x,y] \cap K[p,y]$. Let $\mathfrak{c}$ be the extended ideal $\mathfrak{a}K(p)[x,y]$ and let $\mathfrak{d}$ be the elimination ideal $\mathfrak{c} \cap K(p)[y]$. Then, the extension of $\mathfrak{b}$ coincides with $\mathfrak{d}$, i.e. $\mathfrak{b}^e = \mathfrak{b}K(p)[y] = \mathfrak{d}$.
- **Lemma 2.4.** Under the assumption that none of the primary components of $\mathfrak{a}$ contain a non-zero polynomial in $K[p]$, we claim that the generators of the contraction $\mathfrak{d}^c$ of the elimination ideal $\mathfrak{d} = \mathfrak{a}K(p)[x,y] \cap K(p)[y]$ to $K[p,y]$ generate as well $\mathfrak{b}$, the elimination ideal $\mathfrak{a}K[p,x,y] \cap K[p,y]$

Dana-Picard, T., Recio T. : "From Thales theorem to octic curves". Maple Trans., Vol. 4, No. 3, Article 18022. October 2024 https://doi.org/10.5206/mt.v4i3.18022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GeoGebra Classic
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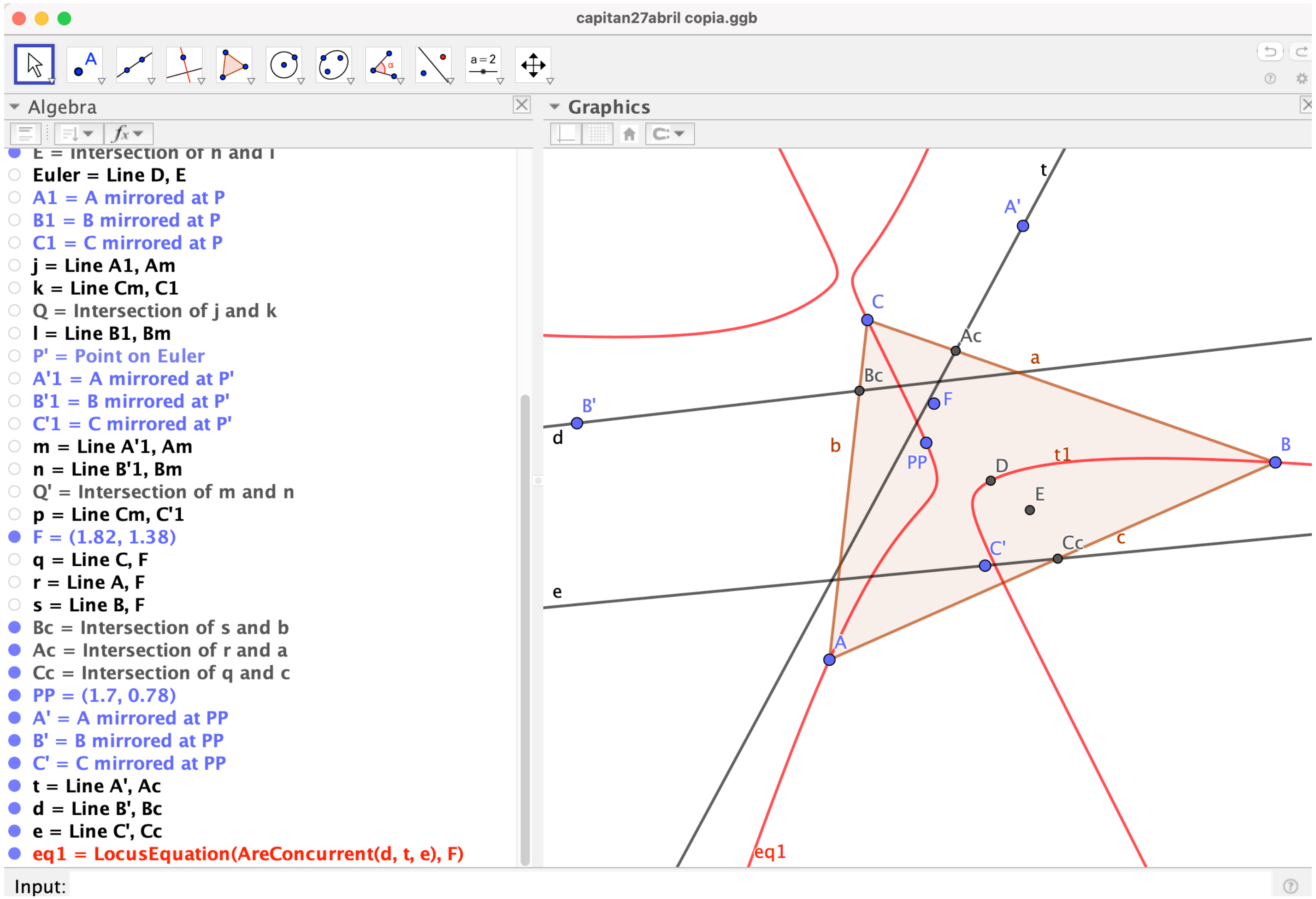
f = Segment(A, C)
→ 5.4
g = Segment(C, B)
→ 5.01
eq1 : LocusEquation(f ≟ 2 g, C)
→ 15000x² + 10900x + 15000y² - 32500y = 288429
Input...
C
g
f
B
A
eq1

Given a triangle ABC, we consider a general point F and the feet of the lines r=AF, s=BF, q=CF, (Ac, Bc,Cc, respectively). Next, we consider another arbitrary pont PP and we consider A'=A mirrorred at PP, B' and C' (likewise). Then we define lines t=A'Ac, d=B'Bc, e=C'Cc.

And we wonder where to place F for having the concurrency of the three lines.



See http://garciacapitan.epizy.com and the generalization of a problema by D. Nguyen, from April 2025
https://garciacapitan.blogspot.com/2025/04/generalizing-problem-nguyen085.html

```
> Prepre:=<f1*v - f2*u, c1*v - c2*u + c2 - v,f1*q - f2*p + f2 - q
, c1*q - c2*p,c1*f2 - c1*s - c2*f1 + c2*r + f1*s - f2*r, s,aa1-2*
pp1, aa2-2*pp2,bb1-(2*pp1-1),bb2-2*pp2,cc1-(2*pp1-c1), cc2-(2*pp2
-c2), c2*t-1>;
```

$$Prepre := \langle s, aa1 - 2\,pp1, aa2 - 2\,pp2, bb2 - 2\,pp2, bb1 - 2\,pp1 + 1, cc1 - 2\,pp1 + c1, cc2 - 2\,pp2 + c2, c2\,t - 1, c1\,q - c2\,p, f1\,v - f2\,u, c1\,v - c2\,u + c2 - v, f1\,q - f2\,p + f2 - q, c1\,f2 - c1\,s - c2\,f1 + c2\,r + f1\,s - f2\,r\rangle \quad \textbf{(5.4)}$$

```
> HilbertDimension(Prepre);
```

$$6 \quad \textbf{(5.5)}$$

```
> EliminationIdeal(Prepre,{c1,c2,f1,f2,pp1,pp2});
```

$$\langle 0\rangle \quad \textbf{(5.6)}$$

```
> Condcond:=<f1*v - f2*u, c1*v - c2*u + c2 - v,f1*q - f2*p + f2 -
  q , c1*q - c2*p,c1*f2 - c1*s - c2*f1 + c2*r + f1*s - f2*r, s,aa1
  -2*pp1, aa2-2*pp2,bb1-(2*pp1-1),bb2-2*pp2,cc1-(2*pp1-c1), cc2-(2*
  pp2-c2),aa1*v - aa1*y - aa2*u + aa2*x + u*y - v*x, bb1*q - bb1*y
  - bb2*p + bb2*x + p*y - q*x,cc1*s - cc1*y - cc2*r + cc2*x + r*y -
  s*x,c2*t-1>;
```

```
> HilbertDimension(Condcond);
```

$$5 \tag{5.10}$$

```
> EliminationIdeal(Condcond,{c1,c2,f1,f2,pp1});
```

$$\langle 0\rangle \tag{5.11}$$

```
> EliminationIdeal(Condcond,{c1,c2,f1,f2,pp1,pp2});
```

$$\begin{aligned}\langle &-2\, pp2\, f2^{3}\, c1^{3} + 2\, pp2^{2}\, f2^{2}\, c1^{3} + 4\, pp2\, f1\, f2^{2}\, c1^{2}\, c2 + 2\, f2^{3}\, pp1\, c1^{2}\, c2 \\
&- 6\, pp2\, f2^{2}\, pp1\, c1^{2}\, c2 - 6\, pp2^{2}\, f1\, f2^{2}\, c1^{2} + 6\, pp2\, f2^{3}\, pp1\, c1^{2} - 2\, pp2\, f1^{2}\, f2\, c1\, c2^{2} \\
&- 2\, pp2^{2}\, f1^{2}\, c1\, c2^{2} - 4\, f1\, f2^{2}\, pp1\, c1\, c2^{2} + 4\, pp2\, f1\, f2\, pp1\, c1\, c2^{2} + 4\, f2^{2}\, pp1^{2}\, c1\, c2^{2} \\
&+ 8\, pp2^{2}\, f1^{2}\, f2\, c1\, c2 - 4\, pp2\, f1\, f2^{2}\, pp1\, c1\, c2 - 4\, f2^{3}\, pp1^{2}\, c1\, c2 + 2\, f1^{2}\, f2\, pp1\, c2^{3} \\
&+ 2\, pp2\, f1^{2}\, pp1\, c2^{3} - 4\, f1\, f2\, pp1^{2}\, c2^{3} - 2\, pp2^{2}\, f1^{3}\, c2^{2} - 2\, pp2\, f1^{2}\, f2\, pp1\, c2^{2} \\
&+ 4\, f1\, f2^{2}\, pp1^{2}\, c2^{2} - f2^{3}\, c1^{2}\, c2 + pp2\, f2^{2}\, c1^{2}\, c2 + 2\, f1\, f2^{2}\, c1\, c2^{2} + 2\, pp2^{2}\, f1\, c1\, c2^{2} \\
&- 2\, f2^{2}\, pp1\, c1\, c2^{2} - 2\, pp2\, f2\, pp1\, c1\, c2^{2} - 2\, pp2\, f1\, f2^{2}\, c1\, c2 - 8\, pp2^{2}\, f1\, f2\, c1\, c2 \\
&+ 2\, f2^{3}\, pp1\, c1\, c2 + 8\, pp2\, f2^{2}\, pp1\, c1\, c2 + 6\, pp2^{2}\, f1\, f2^{2}\, c1 - 6\, pp2\, f2^{3}\, pp1\, c1 \\
&- f1^{2}\, f2\, c2^{3} - pp2\, f1^{2}\, c2^{3} + 2\, f1\, f2\, pp1\, c2^{3} - 2\, pp2\, f1\, pp1\, c2^{3} + 2\, f2\, pp1^{2}\, c2^{3} \\
&+ 2\, pp2\, f1^{2}\, f2\, c2^{2} + 4\, pp2^{2}\, f1^{2}\, c2^{2} - 2\, f1\, f2^{2}\, pp1\, c2^{2} - 4\, f2^{2}\, pp1^{2}\, c2^{2} \\
&- 4\, pp2^{2}\, f1^{2}\, f2\, c2 + 2\, pp2\, f1\, f2^{2}\, pp1\, c2 + 2\, f2^{3}\, pp1^{2}\, c2 - f2^{3}\, c1\, c2 + pp2\, f2^{2}\, c1\, c2 \\
&+ 2\, pp2\, f2^{3}\, c1 - 2\, pp2^{2}\, f2^{2}\, c1 + pp2\, f1\, c2^{3} - f2\, pp1\, c2^{3} + f1\, f2^{2}\, c2^{2} - 2\, pp2\, f1\, f2\, c2^{2} \\
&- 2\, pp2^{2}\, f1\, c2^{2} + f2^{2}\, pp1\, c2^{2} + 2\, pp2\, f2\, pp1\, c2^{2} - 2\, pp2\, f1\, f2^{2}\, c2 + 4\, pp2^{2}\, f1\, f2\, c2 \\
&- 2\, pp2\, f2^{2}\, pp1\, c2\rangle\end{aligned} \tag{5.12}$$

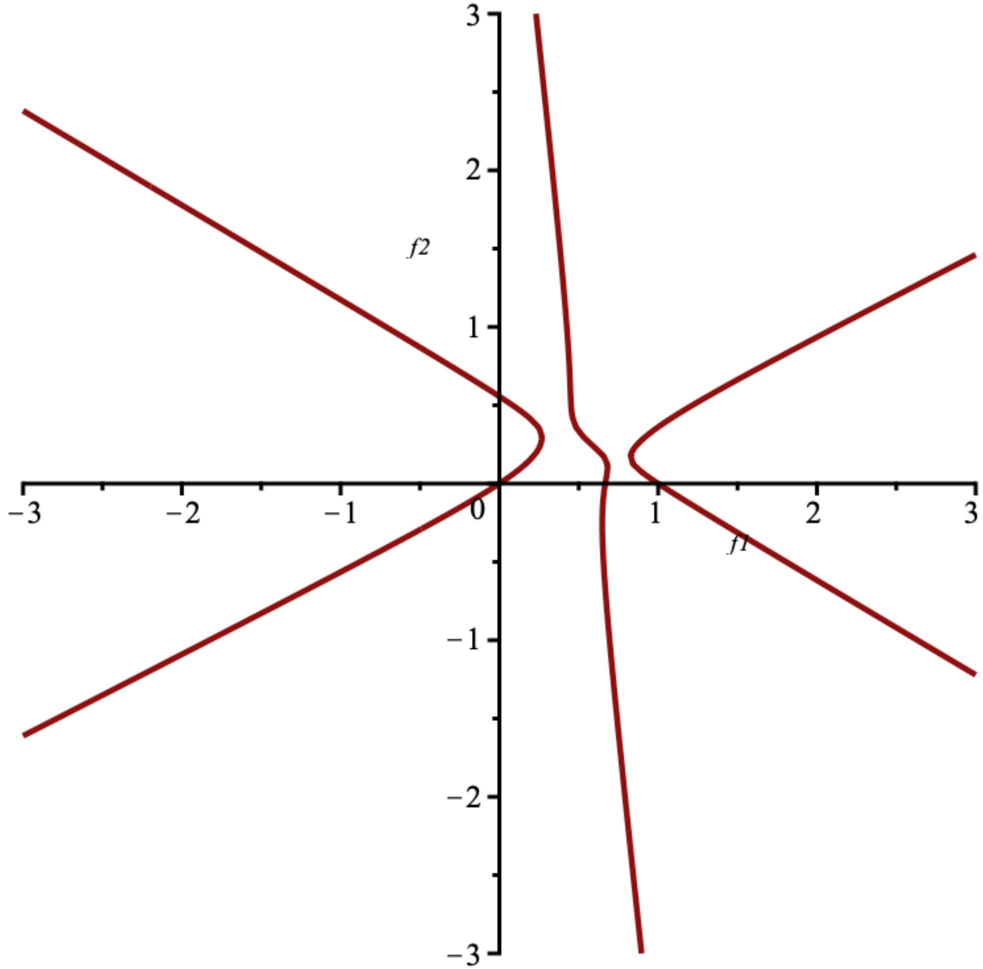
```
> simplify(subs(f1=0,f2=0, -2*c1^3*f2^3*pp2 + 2*c1^3*f2^2*pp2^2 +
4*c1^2*c2*f1*f2^2*pp2 + 2*c1^2*c2*f2^3*pp1 - 6*c1^2*c2*f2^2*pp1*
pp2 - 6*c1^2*f1*f2^2*pp2^2 + 6*c1^2*f2^3*pp1*pp2 - 2*c1*c2^2*
f1^2*f2*pp2 - 2*c1*c2^2*f1^2*pp2^2 - 4*c1*c2^2*f1*f2^2*pp1 + 4*
c1*c2^2*f1*f2*pp1*pp2 + 4*c1*c2^2*f2^2*pp1^2 + 8*c1*c2*f1^2*f2*
pp2^2 - 4*c1*c2*f1*f2^2*pp1*pp2 - 4*c1*c2*f2^3*pp1^2 + 2*c2^3*
f1^2*f2*pp1 + 2*c2^3*f1^2*pp1*pp2 - 4*c2^3*f1*f2*pp1^2 - 2*c2^2*
f1^3*pp2^2 - 2*c2^2*f1^2*f2*pp1*pp2 + 4*c2^2*f1*f2^2*pp1^2 -
c1^2*c2*f2^3 + c1^2*c2*f2^2*pp2 + 2*c1*c2^2*f1*f2^2 + 2*c1*c2^2*
f1*pp2^2 - 2*c1*c2^2*f2^2*pp1 - 2*c1*c2^2*f2*pp1*pp2 - 2*c1*c2*
f1*f2^2*pp2 - 8*c1*c2*f1*f2*pp2^2 + 2*c1*c2*f2^3*pp1 + 8*c1*c2*
f2^2*pp1*pp2 + 6*c1*f1*f2^2*pp2^2 - 6*c1*f2^3*pp1*pp2 - c2^3*
f1^2*f2 - c2^3*f1^2*pp2 + 2*c2^3*f1*f2*pp1 - 2*c2^3*f1*pp1*pp2 +
2*c2^3*f2*pp1^2 + 2*c2^2*f1^2*f2*pp2 + 4*c2^2*f1^2*pp2^2 - 2*
c2^2*f1*f2^2*pp1 - 4*c2^2*f2^2*pp1^2 - 4*c2*f1^2*f2*pp2^2 + 2*c2*
f1*f2^2*pp1*pp2 + 2*c2*f2^3*pp1^2 - c1*c2*f2^3 + c1*c2*f2^2*pp2 +
2*c1*f2^3*pp2 - 2*c1*f2^2*pp2^2 + c2^3*f1*pp2 - c2^3*f2*pp1 +
c2^2*f1*f2^2 - 2*c2^2*f1*f2*pp2 - 2*c2^2*f1*pp2^2 + c2^2*f2^2*pp1
+ 2*c2^2*f2*pp1*pp2 - 2*c2*f1*f2^2*pp2 + 4*c2*f1*f2*pp2^2 - 2*c2*
f2^2*pp1*pp2));
```

$$0 \tag{5.14}$$

```
> simplify(subs(f1=1,f2=0, -2*c1^3*f2^3*pp2 + 2*c1^3*f2^2*pp2^2 +
4*c1^2*c2*f1*f2^2*pp2 + 2*c1^2*c2*f2^3*pp1 - 6*c1^2*c2*f2^2*pp1*
pp2 - 6*c1^2*f1*f2^2*pp2^2 + 6*c1^2*f2^3*pp1*pp2 - 2*c1*c2^2*
f1^2*f2*pp2 - 2*c1*c2^2*f1^2*pp2^2 - 4*c1*c2^2*f1*f2^2*pp1 + 4*
c1*c2^2*f1*f2*pp1*pp2 + 4*c1*c2^2*f2^2*pp1^2 + 8*c1*c2*f1^2*f2*
pp2^2 - 4*c1*c2*f1*f2^2*pp1*pp2 - 4*c1*c2*f2^3*pp1^2 + 2*c2^3*
f1^2*f2*pp1 + 2*c2^3*f1^2*pp1*pp2 - 4*c2^3*f1*f2*pp1^2 - 2*c2^2*
f1^3*pp2^2 - 2*c2^2*f1^2*f2*pp1*pp2 + 4*c2^2*f1*f2^2*pp1^2 -
c1^2*c2*f2^3 + c1^2*c2*f2^2*pp2 + 2*c1*c2^2*f1*f2^2 + 2*c1*c2^2*
f1*pp2^2 - 2*c1*c2^2*f2^2*pp1 - 2*c1*c2^2*f2*pp1*pp2 - 2*c1*c2*
f1*f2^2*pp2 - 8*c1*c2*f1*f2*pp2^2 + 2*c1*c2*f2^3*pp1 + 8*c1*c2*
f2^2*pp1*pp2 + 6*c1*f1*f2^2*pp2^2 - 6*c1*f2^3*pp1*pp2 - c2^3*
f1^2*f2 - c2^3*f1^2*pp2 + 2*c2^3*f1*f2*pp1 - 2*c2^3*f1*pp1*pp2 +
2*c2^3*f2*pp1^2 + 2*c2^2*f1^2*f2*pp2 + 4*c2^2*f1^2*pp2^2 - 2*
c2^2*f1*f2^2*pp1 - 4*c2^2*f2^2*pp1^2 - 4*c2*f1^2*f2*pp2^2 + 2*c2*
f1*f2^2*pp1*pp2 + 2*c2*f2^3*pp1^2 - c1*c2*f2^3 + c1*c2*f2^2*pp2 +
2*c1*f2^3*pp2 - 2*c1*f2^2*pp2^2 + c2^3*f1*pp2 - c2^3*f2*pp1 +
c2^2*f1*f2^2 - 2*c2^2*f1*f2*pp2 - 2*c2^2*f1*pp2^2 + c2^2*f2^2*pp1
+ 2*c2^2*f2*pp1*pp2 - 2*c2*f1*f2^2*pp2 + 4*c2*f1*f2*pp2^2 - 2*c2*
f2^2*pp1*pp2));
```

$$0 \tag{5.15}$$

```
> with(plots, implicitplot):implicitplot(216*f1^3 - 36*f1^2*f2 -
  702*f1*f2^2 - 75*f2^3 - 360*f1^2 + 360*f1*f2 + 430*f2^2 + 144*f1
  - 216*f2,f1=-3..3,f2=-3..3);
```



```
> hh:=<f1*v - f2*u, c1*v - c2*u + c2 - v,f1*q - f2*p + f2 - q ,
  c1*q - c2*p,c1*f2 - c1*s - c2*f1 + c2*r + f1*s - f2*r, s,aa1-2*
  pp1, aa2-2*pp2,bb1-(2*pp1-1),bb2-2*pp2,cc1-(2*pp1-c1), cc2-(2*pp2
  -c2),aa1*v - aa1*y - aa2*u + aa2*x + u*y - v*x, bb1*q - bb1*y -
  bb2*p + bb2*x + p*y - q*x,c2*t-1,-2*c1^3*f2^3*pp2 + 2*c1^3*f2^2*
  pp2^2 + 4*c1^2*c2*f1*f2^2*pp2 + 2*c1^2*c2*f2^3*pp1 - 6*c1^2*c2*
  f2^2*pp1*pp2 - 6*c1^2*f1*f2^2*pp2^2 + 6*c1^2*f2^3*pp1*pp2 - 2*c1*
  c2^2*f1^2*f2*pp2 - 2*c1*c2^2*f1^2*pp2^2 - 4*c1*c2^2*f1*f2^2*pp1 +
  4*c1*c2^2*f1*f2*pp1*pp2 + 4*c1*c2^2*f2^2*pp1^2 + 8*c1*c2*f1^2*f2*
  pp2^2 - 4*c1*c2*f1*f2^2*pp1*pp2 - 4*c1*c2*f2^3*pp1^2 + 2*c2^3*
  f1^2*f2*pp1 + 2*c2^3*f1^2*pp1*pp2 - 4*c2^3*f1*f2*pp1^2 - 2*c2^2*
  f1^3*pp2^2 - 2*c2^2*f1^2*f2*pp1*pp2 + 4*c2^2*f1*f2^2*pp1^2 -
  c1^2*c2*f2^3 + c1^2*c2*f2^2*pp2 + 2*c1*c2^2*f1*f2^2 + 2*c1*c2^2*
  f1*pp2^2 - 2*c1*c2^2*f2^2*pp1 - 2*c1*c2^2*f2*pp1*pp2 - 2*c1*c2*
  f1*f2^2*pp2 - 8*c1*c2*f1*f2*pp2^2 + 2*c1*c2*f2^3*pp1 + 8*c1*c2*
  f2^2*pp1*pp2 + 6*c1*f1*f2^2*pp2^2 - 6*c1*f2^3*pp1*pp2 - c2^3*
  f1^2*f2 - c2^3*f1^2*pp2 + 2*c2^3*f1*f2*pp1 - 2*c2^3*f1*pp1*pp2 +
  2*c2^3*f2*pp1^2 + 2*c2^2*f1^2*f2*pp2 + 4*c2^2*f1^2*pp2^2 - 2*
  c2^2*f1*f2^2*pp1 - 4*c2^2*f2^2*pp1^2 - 4*c2*f1^2*f2*pp2^2 + 2*c2*
  f1*f2^2*pp1*pp2 + 2*c2*f2^3*pp1^2 - c1*c2*f2^3 + c1*c2*f2^2*pp2 +
  2*c1*f2^3*pp2 - 2*c1*f2^2*pp2^2 + c2^3*f1*pp2 - c2^3*f2*pp1 +
  c2^2*f1*f2^2 - 2*c2^2*f1*f2*pp2 - 2*c2^2*f1*pp2^2 + c2^2*f2^2*pp1
  + 2*c2^2*f2*pp1*pp2 - 2*c2*f1*f2^2*pp2 + 4*c2*f1*f2*pp2^2 - 2*c2*
  f2^2*pp1*pp2>;
```

$$hh := \langle s, aa1 - 2\,pp1, aa2 - 2\,pp2, bb2 - 2\,pp2, bb1 - 2\,pp1 + 1, cc1 - 2\,pp1 + c1, cc2 \qquad \textbf{(6.4)}$$

$$- 2\,pp2 + c2, c2\,t - 1, c1\,q - c2\,p, f1\,v - f2\,u, c1\,v - c2\,u + c2 - v, f1\,q - f2\,p + f2$$

$$- q, aa1\,v - aa1\,y - aa2\,u + aa2\,x + u\,y - v\,x, bb1\,q - bb1\,y - bb2\,p + bb2\,x + p\,y$$

$$- q\,x, c1\,f2 - c1\,s - c2\,f1 + c2\,r + f1\,s - f2\,r, -2\,c1^3\,f2^3\,pp2 + 2\,c1^3\,f2^2\,pp2^2$$

$$+ 4\,c1^2\,c2\,f1\,f2^2\,pp2 + 2\,c1^2\,c2\,f2^3\,pp1 - 6\,c1^2\,c2\,f2^2\,pp1\,pp2 - 6\,c1^2\,f1\,f2^2\,pp2^2$$

$$+ 6\,c1^2\,f2^3\,pp1\,pp2 - 2\,c1\,c2^2\,f1^2\,f2\,pp2 - 2\,c1\,c2^2\,f1^2\,pp2^2 - 4\,c1\,c2^2\,f1\,f2^2\,pp1$$

$$+ 4\,c1\,c2^2\,f1\,f2\,pp1\,pp2 + 4\,c1\,c2^2\,f2^2\,pp1^2 + 8\,c1\,c2\,f1^2\,f2\,pp2^2$$

$$- 4\,c1\,c2\,f1\,f2^2\,pp1\,pp2 - 4\,c1\,c2\,f2^3\,pp1^2 + 2\,c2^3\,f1^2\,f2\,pp1 + 2\,c2^3\,f1^2\,pp1\,pp2$$

$$- 4\,c2^3\,f1\,f2\,pp1^2 - 2\,c2^2\,f1^3\,pp2^2 - 2\,c2^2\,f1^2\,f2\,pp1\,pp2 + 4\,c2^2\,f1\,f2^2\,pp1^2$$

$$- c1^2\,c2\,f2^3 + c1^2\,c2\,f2^2\,pp2 + 2\,c1\,c2^2\,f1\,f2^2 + 2\,c1\,c2^2\,f1\,pp2^2 - 2\,c1\,c2^2\,f2^2\,pp1$$

$$- 2\,c1\,c2^2\,f2\,pp1\,pp2 - 2\,c1\,c2\,f1\,f2^2\,pp2 - 8\,c1\,c2\,f1\,f2\,pp2^2 + 2\,c1\,c2\,f2^3\,pp1$$

- The ideal hh is of dimension 5, and C, PP, f1 can be considered (this is human intuition) as the "ruling" independent variables (it is reasonable, since there is an equation (the condition) holding between C, F, PP).
- Now we add the negation of the thesis

$$(cc1*s - cc1*y - cc2*r + cc2*x + r*y - s*x)*g-1$$

with g dumb variable, and check (after 8 minutes computation) that the elimination of the new ideal ll over the C, PP, f1 variables is not zero .

```
> factor(-8*c1^4*f1^2*pp2^4 + 8*c1^3*c2*f1^3*pp2^3 + 16*c1^3*c2*
  f1^2*pp1*pp2^3 - 16*c1^2*c2^2*f1^3*pp1*pp2^2 - 8*c1^2*c2^2*f1^2*
  pp1^2*pp2^2 - 8*c1^2*c2*f1^4*pp2^3 + 8*c1^2*f1^4*pp2^4 + 8*c1*
  c2^3*f1^3*pp1^2*pp2 + 16*c1*c2^2*f1^4*pp1*pp2^2 - 16*c1*c2*f1^4*
  pp1*pp2^3 - 8*c2^3*f1^4*pp1^2*pp2 + 8*c2^2*f1^4*pp1^2*pp2^2 + 8*
  c1^4*f1*pp2^4 - 20*c1^3*c2*f1^2*pp2^3 - 16*c1^3*c2*f1*pp1*pp2^3 +
  16*c1^3*f1^2*pp2^4 + 8*c1^2*c2^2*f1^3*pp2^2 + 32*c1^2*c2^2*f1^2*
  pp1*pp2^2 + 8*c1^2*c2^2*f1*pp1^2*pp2^2 + 4*c1^2*c2*f1^3*pp2^3 -
  24*c1^2*c2*f1^2*pp1*pp2^3 - 16*c1^2*f1^3*pp2^4 - 8*c1*c2^3*f1^3*
  pp1*pp2 - 12*c1*c2^3*f1^2*pp1^2*pp2 - 8*c1*c2^2*f1^4*pp2^2 - 16*
  c1*c2^2*f1^3*pp1*pp2^2 + 8*c1*c2^2*f1^2*pp1^2*pp2^2 + 16*c1*c2*
  f1^4*pp2^3 + 32*c1*c2*f1^3*pp1*pp2^3 - 8*c1*f1^4*pp2^4 + 8*c2^3*
  f1^4*pp1*pp2 + 12*c2^3*f1^3*pp1^2*pp2 - 16*c2^2*f1^4*pp1*pp2^2 -
  16*c2^2*f1^3*pp1^2*pp2^2 + 8*c2*f1^4*pp1*pp2^3 + 12*c1^3*c2*f1*
  pp2^3 - 16*c1^3*f1*pp2^4 - 14*c1^2*c2^2*f1^2*pp2^2 - 16*c1^2*
  c2^2*f1*pp1*pp2^2 + 20*c1^2*c2*f1^2*pp2^3 + 24*c1^2*c2*f1*pp1*
  pp2^3 + 2*c1*c2^3*f1^3*pp2 + 12*c1*c2^3*f1^2*pp1*pp2 + 4*c1*c2^3*
  f1*pp1^2*pp2 + 8*c1*c2^2*f1^3*pp2^2 - 12*c1*c2^2*f1^2*pp1*pp2^2 -
  8*c1*c2^2*f1*pp1^2*pp2^2 - 28*c1*c2*f1^3*pp2^3 - 8*c1*c2*f1^2*
  pp1*pp2^3 + 16*c1*f1^3*pp2^4 - 2*c2^3*f1^4*pp2 - 12*c2^3*f1^3*
  pp1*pp2 - 4*c2^3*f1^2*pp1^2*pp2 + 6*c2^2*f1^4*pp2^2 + 28*c2^2*
  f1^3*pp1*pp2^2 + 8*c2^2*f1^2*pp1^2*pp2^2 - 4*c2*f1^4*pp2^3 - 16*
  c2*f1^3*pp1*pp2^3 + 6*c1^2*c2^2*f1*pp2^2 - 16*c1^2*c2*f1*pp2^3 +
  8*c1^2*f1*pp2^4 - 3*c1*c2^3*f1^2*pp2 - 4*c1*c2^3*f1*pp1*pp2 + 4*
  c1*c2^2*f1^2*pp2^2 + 12*c1*c2^2*f1*pp1*pp2^2 + 8*c1*c2*f1^2*pp2^3
  - 8*c1*c2*f1*pp1*pp2^3 - 8*c1*f1^2*pp2^4 + 3*c2^3*f1^3*pp2 + 4*
  c2^3*f1^2*pp1*pp2 - 10*c2^2*f1^3*pp2^2 - 12*c2^2*f1^2*pp1*pp2^2 +
  8*c2*f1^3*pp2^3 + 8*c2*f1^2*pp1*pp2^3 + c1*c2^3*f1*pp2 - 4*c1*
  c2^2*f1*pp2^2 + 4*c1*c2*f1*pp2^3 - c2^3*f1^2*pp2 + 4*c2^2*f1^2*
  pp2^2 - 4*c2*f1^2*pp2^3);
```

$$-f1\,pp2\,(f1-1)\,(c1-f1)\,(2\,c1\,pp2-2\,pp1\,c2+c2-2\,pp2)\,(2\,c1\,pp2-2\,pp1\,c2+c2)\,(2\,c1\,pp2-2\,c2\,f1+2\,f1\,pp2+c2-2\,pp2) \tag{6.9}$$

Thank you for your attention

Any questions or comments?