

Using Differential Equations with Maple in General Relativity

Maple Application: The advance of perihelion of Mercury

Bahia SI LAKHAL

Faculty of Physics, USTHB, Algiers, ALGERIA

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Outline

- 1 Introduction
- 2 What is the perihelion?
- 3 Background: Relativistic Motion Near the Sun
- 4 The Schwarzschild Metric
- 5 The Euler–Lagrange Equations
- 6 Orbital Equation
- 7 Application to Mercury planet
- 8 Numerical Results
- 9 Conclusion

Introduction

Tests of general relativity provide crucial observational evidence supporting Einstein's theory. The first three classical tests, proposed by Albert Einstein in 1915, concern:

- the anomalous precession of the perihelion of Mercury,
- the deflection of light by a gravitational field, and
- the gravitational redshift of light.

The advance (or precession) of Mercury's perihelion is one of the most famous phenomena that confirmed Einstein's general theory of relativity. In this work, we aim to use *Maple* software to solve the orbital equation of a planet and obtain the solution that describes the relativistic advance of its perihelion.

What is the perihelion?

For a planet orbiting the Sun, the **perihelion** is the point of closest approach to the Sun.

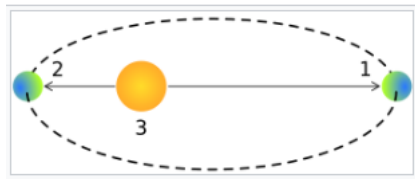


Figure: 1. Planet at aphelion 2. Planet at perihelion 3. Sun. (Photo taken from Wikipedia)

- In Newtonian mechanics, the orbit of a planet around the Sun is an **ellipse** (according to Kepler's first law) that remains **fixed in space**, the perihelion does not move.
- However, observations, particularly for Mercury, reveal that this elliptical orbit slowly rotates over time: the perihelion advances slightly after each revolution. This phenomenon is known as the **advance (or precession) of the perihelion**.

Background: Relativistic Motion Near the Sun

- In general relativity, the Sun's gravity is not described as a force acting in flat space. Instead, the Sun's mass curves the surrounding spacetime, and Mercury follows a geodesic, the natural path of motion, within this curved geometry.
- The Schwarzschild metric provides the exact solution of Einstein's field equations describing the spacetime outside a static, spherically symmetric, and non-rotating mass M .

We define a quantity [1]

$$2K \equiv g_{ab}\dot{x}^a\dot{x}^b = \alpha, \quad (1)$$

where α is a constant characterising the nature of the geodesic:

$$2K = \alpha = \begin{cases} 0, & \text{for a null (lightlike) geodesic,} \\ +1, & \text{for a timelike geodesic,} \\ -1, & \text{for a spacelike geodesic,} \end{cases}$$

The Schwarzschild Metric

Here, the dot denotes differentiation with respect to an affine parameter, which is chosen to be the proper time τ for timelike trajectories.

A convenient form of the geodesic equation can then be written as

$$\frac{\partial K}{\partial x^a} - \frac{d}{d\tau} \left(\frac{\partial K}{\partial \dot{x}^a} \right) = 0, \quad a = 0, \dots, 3 \quad (2)$$

We can study the orbit of a test particle (e.g., a planet) around the Sun by considering geodesic motion in the Schwarzschild metric. The quantity $2K$ is given by

$$2K = \left(1 - \frac{R_s}{r} \right) c^2 \dot{t}^2 - \left(1 - \frac{R_s}{r} \right)^{-1} \dot{r}^2 - r^2 \dot{\theta}^2 - r^2 \sin^2(\theta) \dot{\phi}^2 = c^2, \quad (3)$$

where $R_s = 2\frac{GM}{c^2}$ is the Schwarzschild radius of the central mass (the Sun).

The Euler–Lagrange Equations

Applying the Euler–Lagrange equations to the coordinates $x^a = (t, r, \theta, \phi)$ gives the following relations for $a = 0, 2$ and 3 .

$$\frac{d}{d\tau} \left[\left(1 - \frac{R_s}{r}\right) \dot{t} \right] = 0. \quad (4)$$

$$\frac{d}{d\tau} \left(r^2 \dot{\theta} \right) - r^2 \sin \theta \cos \theta \dot{\phi}^2 = 0. \quad (5)$$

$$\frac{d}{d\tau} \left(r^2 \sin^2 \theta \dot{\phi} \right) = 0. \quad (6)$$

Motion in the Equatorial Plane

In Newtonian mechanics, a planet moves in a fixed plane. We now examine whether this remains true in general relativity.

Assume that the motion takes place in the equatorial plane, such that

$$\theta(\tau_0) = \frac{\pi}{2} \text{ and } \dot{\theta}(\tau_0) = 0. \quad (7)$$

From Eq. (5), it follows that

The Euler–Lagrange Equations

$$\frac{d}{d\tau} (r^2 \dot{\theta}) = 0 \implies r^2 \dot{\theta} \Big|_{\tau} - r^2 \dot{\theta} \Big|_{\tau_0} = 0 \implies r^2 \dot{\theta} \Big|_{\tau} = 0. \quad (8)$$

Since $\dot{\theta}(\tau_0) = 0$, the constant is zero, and therefore $\dot{\theta} = 0$.

Hence, the motion is **confined to the equatorial plane** ($\theta = \frac{\pi}{2}$) throughout the trajectory, just as in the Newtonian case.

This proves that geodesic trajectories in Schwarzschild spacetime must be planar. Then (6) gives

$$r^2 \dot{\phi} = h, \quad (9)$$

where h is a constant. Equation (9) expresses the conservation of angular momentum, with h representing the angular momentum per unit mass of the planet.

Similarly, equation (4) gives

$$\left(1 - \frac{R_s}{r}\right) \dot{t} = E, \quad (10)$$

Change of variable

where E is another constant of motion, associated with the conservation of energy.

Substituting these two integrals of motion into Eq. (3), we obtain

$$E^2 c^2 \left(1 - \frac{R_s}{r}\right)^{-1} - \left(1 - \frac{R_s}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 = c^2. \quad (11)$$

To study the **shape of the orbit** $r(\phi)$, it is convenient to introduce the variable

$$u = \frac{1}{r}. \quad (12)$$

Since we are not interested in the proper time τ but rather in how r varies with ϕ , we use ϕ as the independent variable in the orbital equation.

From (9), we have

$$r = \frac{1}{u} \implies \dot{r} = -\frac{\dot{u}}{u^2}, \quad (13)$$

Orbital Equation

and

$$\dot{\phi} = hu^2 \implies \frac{d\phi}{du} \dot{u} = hu^2 \implies \dot{u} = hu^2 \frac{du}{d\phi}. \quad (14)$$

Combining these expressions gives

$$\dot{r} = -h \frac{du}{d\phi}. \quad (15)$$

Substituting these relations into Eq. (11) yields

$$E^2 c^2 (1 - R_s u)^{-1} - (1 - R_s u)^{-1} h^2 \left(\frac{du}{d\phi} \right)^2 - h^2 u^2 = c^2. \quad (16)$$

After straightforward simplification, we obtain the first-order differential equation governing the orbit:

Orbital Equation

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{(E^2 - 1)c^2}{h^2} + \frac{R_s c^2}{h^2} u + R_s u^3. \quad (17)$$

Equation (17) represents **the relativistic orbital equation for a planet** moving in the Schwarzschild geometry.

We now apply an **approximate method** to solve the orbital equation. Differentiating Eq. (17) with respect to ϕ , we obtain

$$\frac{d^2 u}{d\phi^2} + u = \frac{R_s c^2}{2h^2} + \frac{3}{2} R_s u^2. \quad (18)$$

This is **the relativistic form of Binet's equation**. It differs from the classical (Newtonian) version by the presence of the additional term $\frac{3}{2} R_s u^2$ on the right-hand side, which represents the **general relativistic correction** responsible for the **precession of the perihelion**.

Orbital Equation

If we neglect this second term, Eq. (18) reduces to **the Newtonian orbital equation**:

$$\frac{d^2 u}{d\phi^2} + u = \frac{R_s c^2}{2h^2}. \quad (19)$$

Its solution corresponds to a **Keplerian ellipse**:

$$u(\phi) = \frac{R_s c^2}{2h^2} (1 + e \cos \phi). \quad (20)$$

e is the **excentricity** of the ellipse.

Since the Schwarzschild radius R_s of the Sun is extremely small compared to the orbital radius of a planet, we expect that the general relativistic corrections to the Newtonian motion are very weak. Consequently, the orbit remains **approximately elliptical**, but with a subtle **precession of the perihelion** after each revolution.

In Newtonian mechanics, the orbit of a planet around the Sun is a closed ellipse. However, in **general relativity**, the additional term $\frac{3}{2}R_s u^2$ in Eq. (18) slightly perturbs this motion, causing the ellipse to rotate slowly within its plane. This means that the perihelion advances by a small angle after each orbit.

By solving Eq. (18) perturbatively, treating the relativistic term as a small correction to the Newtonian solution, one finds that the angular shift of the perihelion per revolution is given by [2]:

$$\Delta\phi = \frac{6\pi GM}{a(1 - e^2)c^2}, \quad (21)$$

where

- a is the semi-major axis of the orbit,
- e is the excentricity of the orbit, and
- M is the mass of the central body (the Sun in this case).

Application to Mercury planet

Applying to the planet Mercury, we have

- $a = 0.57909 \times 10^{11} \text{ m}$
- $e = 0.206,$
- $M_{\odot} = 1.989 \times 10^{30} \text{ kg}.$

Substituting in (21), we find

$$\Delta\phi = 5.015568339 \times 10^{-7} \text{ radians/revolution} \quad (22)$$

$$= 0.1034535232 \text{ arcsecond/revolution}, \quad (23)$$

Since Mercury completes approximately

$$100 \times \frac{365.25}{87.97} \approx 415 \text{ revolutions per century}, \quad (24)$$

the cumulative precession is $\Delta\phi \approx 43''$ per century.

This value agrees remarkably well with the **observed anomalous precession of Mercury's perihelion**, providing one of the most celebrated confirmations of **Einstein's general theory of relativity**.

Application to Mercury planet

Equation (18) can be solved numerically using the *dsolve* command in *Maple* to obtain the function $u(\phi) = \frac{1}{r(\phi)}$, which describes the trajectory of the planet in the Schwarzschild spacetime.

The goal of this numerical integration is to determine the shape of the orbit and, in particular, the position of the perihelion, corresponding to the minimum value of r (or equivalently, the maximum of u).

To locate this point, we note that at the perihelion,

$$\frac{dr}{d\phi} = 0. \quad (25)$$

Since $r(\phi) = \frac{1}{u(\phi)}$, this condition is equivalent to

$$\frac{du}{d\phi} = 0. \quad (26)$$

Numerical Results with *Maple*

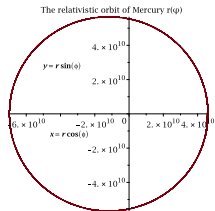
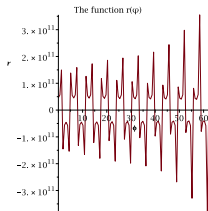
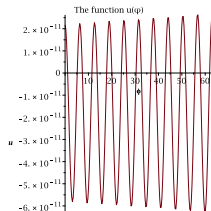
Thus, in the numerical solution, we can identify the angular position $\phi = 0$ where this derivative vanishes at the point of closest approach.

In practice, this is implemented in a simple *Maple* worksheet as follows:

- 1 Define the differential equation (18) for $u(\phi)$.
- 2 Specify appropriate initial conditions, such as $u(0) = u_0$ and $\left. \frac{du}{d\phi} \right|_0 = 0$, corresponding to the perihelion.
- 3 Use the *dsolve* command with the numeric option to integrate the equation over several orbital periods.
- 4 Extract the successive maxima of $u(\phi)$ (or minima of $r(\phi)$) to determine the angular shift of the perihelion after each revolution.

This numerical approach provides a direct and illustrative way to confirm the analytical prediction of the **relativistic advance of the perihelion**.

Numerical Results with *Maple*



The plots of $r(\phi)$ and $u(\phi)$ exhibit small periodic variations. Over short timescales, the precession of the orbit is extremely small and therefore not easily visible in the plots.

Conclusion

- Using *Maple 2018*, we have numerically and analytically studied one of the most famous tests of general relativity: the **relativistic precession of Mercury's perihelion**.
- The computed results are in excellent agreement with both the analytical formula and astronomical observations, beautifully illustrating how the curvature of spacetime explains phenomena that cannot be accounted for by Newtonian mechanics.

References

- [1] R. A. d'Inverno, **Introducing Einstein's Relativity**, (Southampton U.) (1992)
- [2] B. F. Schutz, **a first course in general relativity**, Cambridge university press, 1985.